Math 317 Test 2 Solutions

1. a) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; x+y=0 \right\}$ is a subspace

i) $0+0=0$ so $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ and $W \neq \emptyset$

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in W$. Then $x+y=0$ and $u+v=0$.

ii) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} x+u \\ y+v \\ z+w \end{bmatrix}$

$(x+u)+(y+v) = (x+y)+(u+v) = 0+0=0$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in W$ and W is closed under $+$

iii) Let $c \in \mathbb{R}$. $c\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$

$cx+cy = c(x+y) = c0 = 0$ so $c\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in W$

and W is closed under scalar mult.

b) W is not a subspace because

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in W$ since $1 \cdot 0 = 0$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in W$ since $0 \cdot 1 = 0$

but $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin W$ since $1 \cdot 1 = 1 \neq 0$

so W is not closed under $+$

2. a) $[r-s+t,\ r-s,\ r-s-t] = r[1,1,1] + s[-1,-1,-1] + t[1,0,-1]$

so $S = \{[1,1,1], [-1,-1,-1], [1,0,-1]\}$ spans V, Shrink S to a basis:

2nd is multiple of 1st - omit it, 3rd & 1st not multiples of each other so $\{[1,1,1], [1,0,-1]\}$ is a basis

(or use RREF $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ to select 1st & 3rd)

b) $p(x) = ax^2 + bx + c \quad p(1) = 0 \quad a+b+c = 0$

$c = -a-b \quad p(x) = ax^2 + bx + (-a-b)$

$= a(x^2-1) + b(x-1)$

basis is $\{x^2-1, x-1\}$

3. a) $\begin{vmatrix} x-2 & -3 \\ 1 & x-6 \end{vmatrix} = (x-2)(x-6)+3 = x^2-8x+15 = (x-3)(x-5)$

$\lambda = 3 \quad A - 3I = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$x_1 - 3x_2 = 0 \quad \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x_1 = 3x_2$

$\lambda = 5 \quad A - 5I = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 - x_2 = 0 \\ x_1 = x_2 \end{matrix} \quad \vec{v}_5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $\begin{vmatrix} 8g & 8h & 8i \\ d+a & e+b & f+c \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d+a & e+b & f+c \\ 8g & 8h & 8i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ 8g & 8h & 8i \end{vmatrix} = -8\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -8 \cdot -5 = 40$

(4)

	1. Let A be a nonsingular matrix Let λ be an eigenvalue of A	hypotheses
	2. $\exists \vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda \vec{v}$	def eigenvalue
*	3. $A\vec{x} = \vec{0}$ has no nozero solutions	H 11
*	4. $0\vec{v} = \vec{0}$	Thm 4.1
*	5. $\lambda \neq 0$	#2, #3, #4
	6. $A^{-1}A\vec{v} = A^{-1}\lambda\vec{v}$	mult both sides #2 by A^{-1} exists by def nonsingular
	7. $\vec{v} = \lambda A^{-1}\vec{v}$	def A^{-1}, matrix arith
	8. $\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$	arith, $\lambda \neq 0$ by #5
	9. $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}	def eigenvalue

(5)

1. Let A_i be $n \times n$ matrices	hypothesis
2. $\det A_1 = \det A_1$	anything equals itself
3. Assume $\det(A_1 \dots A_k) = (\det A_1)\dots(\det A_k)$	induction hypothesis
4. $\det(A_1 \dots A_k A_{k+1}) = \det(A_1 \dots A_k)\det A_{k+1}$	Thm 3.7
5. $= (\det A_1)\dots(\det A_k)\det A_{k+1}$	# 3
6. For any r, $\det(A_1 \dots A_r) = (\det A_1)\dots(\det A_r)$	#2, #3 ⇒ #5 proved by induction

(6)

1. Let $\vec{v}, \vec{w}, \vec{z}, \vec{x} \in V$ with $\vec{x}$ a linear combination of $\vec{v}, \vec{w}$	hypothesis
2. $\exists\, a, b, c \in \mathbb{R}$ such that $a\vec{v} + b\vec{w} = \vec{x}$	def linear combination
3. $a\vec{v} + b\vec{w} + 0\vec{z} = \vec{x} + \vec{0}$	#3, Thm 4.1
4. $= \vec{x}$	Def vector space
5. $\vec{x}$ is a linear combination of $\vec{v}, \vec{w}, \vec{z}$	Def linear combination