Techniques for determining the minimum rank of a small graph

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Abstract

The minimum rank of a simple graph G is defined to be the smallest possible rank over all symmetric real matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. Minimum rank is a difficult parameter to compute. However, there are now a number of known reduction techniques and bounds that can be programmed on a computer; we have developed a program using the open-source mathematics software *Sage* to implement several techniques. We have also established several additional strategies for computation of minimum rank. These techniques have been used to determine the minimum ranks of all graphs of order 7.

Keywords. minimum rank, maximum nullity, zero forcing number, Sage program, mathematical software, symmetric matrix, rank, matrix, tree, planar graph, graph.
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1 Introduction

The *minimum rank problem* (for a simple undirected graph, over the real numbers) is to determine the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given graph G. See [8] for a survey of known results and a discussion of the motivation for the minimum rank problem, as well as a lengthy bibliography. A website [2] with extensive information about minimum rank, including online catalogs of minimum rank and other parameters for families of graphs and small graphs was developed at the AIM workshop "Spectra of families of matrices described by graphs, digraphs, and sign patterns" held in October 2006.

Minimum rank is a difficult parameter to compute, but a number of bounds and reduction tools for computation are now known. In addition, the minimum ranks of numerous families of graphs are also known (e.g., see [1]). Prior to the work described in this paper, the minimum ranks of all graphs of order at most six were known. We have developed a program [7] in the free open-source computer mathematics software system *Sage* [15] that utilizes many of the known bounds and reduction techniques to obtain bounds on the minimum rank of a given graph; the algorithm used by the program is described in the Appendix (Section 5), and Section 2 summarizes known results used by the program. We have also established several strategies for producing a matrix that realizes the minimum rank of a graph (see Section 3). The program, these strategies, and

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other known results have been used to determine the minimum ranks of all graphs of order 7, and those minimum ranks are now recorded in a spreadsheet; this is summarized in Section 4. The spreadsheet and program are available for examination on ArXiv [6, 7], and on the AIM workshop webpage [2], in both PDF and original source format (comma separated text and *Sage* source code, repsectively).

A *graph* $G = (V, E)$ means a simple undirected graph (i.e., neither loops nor multiple edges allowed). For standard graph and minimum rank terminology not defined below, see [8] or [1]. The *order* of a graph G will be denoted by $|G|$, and the adjacency matrix of G will be denoted by $\mathcal{A}(G)$.

The set of $n \times n$ real symmetric matrices will be denoted by S_n. For $A \in S_n$, the *graph* of A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}(A)$. The *minimum rank* of a graph G is

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) : A \in S_{|G|}, \mathcal{G}(A) = G\},$$

and the *maximum nullity* of G is

$$\mathrm{M}(G) = \max\{\mathrm{null}(A) : A \in S_{|G|}, \mathcal{G}(A) = G\}.$$

Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$. An *optimal matrix* for G is a matrix A such that $\mathcal{G}(A) = G$ and $\mathrm{rank}(A) = \mathrm{mr}(G)$.

The *length* of a path is the number of edges in the path, the *distance* between two vertices in a connected graph G is the least length of a path between them, and the *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum distance between two vertices.

A *cut-vertex* of a connected graph is a vertex whose deletion disconnects G. A graph is *planar* if it has a drawing in the plane without edge crossings; such a drawing has one infinite face. A graph is *outerplanar* if it has such a drawing in which every vertex is on the boundary of the infinite face. A *polygonal path* is a "path" of cycles built up one cycle at a time by identifying an edge of a new cycle with an edge (that has a vertex of degree 2) of the most recently added cycle.

The *union* of a set of graphs $G_i = (V_i, E_i), i = 1, \ldots, h$, is $\cup_{i=1}^{h} G_i = (\cup_{i=1}^{h} V_i, \cup_{i=1}^{h} E_i)$; note that the union need not be disjoint. A set of cliques $G_i, i = 1, \ldots, h$, of G such that $G = \cup_{i=1}^{h} G_i$ is called a *clique covering* of G. The *clique covering number* of G, denoted by $\mathrm{cc}(G)$, is the minimum number of cliques required to form a clique covering of G.

The zero forcing number $\mathrm{Z}(G)$ was introduced in [1], where it was shown that $\mathrm{M}(G) \leq \mathrm{Z}(G)$. Colin de Verdière introduced the parameter $\mu(G)$ in [5]. It is clear from the definition of $\mu(G)$ that $\mu(G) \leq \mathrm{M}(G)$ (see [13]).

2 Results used by the program for computation of minimum rank

This section summarizes the known minimum rank results used by the program [7]. Note that other results for the computation of minimum rank are known, cf. Section 3 or [8], but are not included here as they were not implemented in the program.

It is well known that if a graph G is disconnected with connected components $G_i, i = 1, \ldots, h$, then $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i)$; therefore the program first breaks the graph into connected components and then implements the results below on each component.

Theorem 2.1. [1] *If T is a tree, then* $\mathrm{mr}(T) = |T| - \mathrm{Z}(T)$.

Note that there are better algorithms for computing the minimum rank of a tree (e.g., [8]), but for a general purpose program that computes the minimum rank of a small graph, the result in Theorem 2.1 is satisfactory for a tree.

Theorem 2.2. Lower bounds for minimum rank
Let G be a connected graph.

1. [1] *For any graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$ and $|G| - \mathrm{Z}(G) \leq \mathrm{mr}(G)$.*

2. [4] *If G contains as an induced subgraph any of the graphs P_4, Dart, $\ltimes$, $K_{3,3,3}$, then $\mathrm{mr}(G) \geq 3$.*

3. [8] $\mathrm{diam}(G) \leq \mathrm{mr}(G)$.

Theorem 2.3. Upper bounds for minimum rank
Let G be a connected graph.

1. [8] $\mathrm{mr}(G) \leq \mathrm{cc}(G)$.

2. [4] *If G does not contain as an induced subgraph any of the graphs P_4, Dart, $\ltimes$, $K_{3,3,3}$, then $\mathrm{mr}(G) \leq 2$.*

3. [9] $\mathrm{mr}(G) = |G| - 1$ *if and only if G is a path. Thus if G is not a path, then* $\mathrm{mr}(G) \leq |G| - 2$.

4. [5], [13]

   (a) *If G is not planar, then $\mathrm{M}(G) \geq \mu(G) \geq 4$, so $\mathrm{mr}(G) \leq |G| - 4$.*

   (b) *If G is not outerplanar, then $\mathrm{M}(G) \geq \mu(G) \geq 3$, so $\mathrm{mr}(G) \leq |G| - 3$.*

The following theorem is a reduction theorem that computes the minimum rank of a graph from the minimum ranks of lower order graphs.

Theorem 2.4. [3] *Let v be a cut-vertex of G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the set of vertices of the ith component of $G - v$, and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then*

$$\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + \min\left\{ \sum_{i=1}^{h} \big(\mathrm{mr}(G_i) - \mathrm{mr}(G_i - v) \big),\ 2 \right\}.$$

3 Additional techniques for computation of minimum rank

This section lists some known results and presents some new techniques for producing low rank matrices, all of which were used to determine by hand the minimum ranks of the order 7 graphs that were not computed by the program (cf. Section 4).

Theorem 3.1. [12] *Let G be 2-connected. Then $\mathrm{mr}(G) = |G| - 2$ if and only if G is a polygonal path.*

The next observation is well known.

Observation 3.2. *If $G = \cup_{i=1}^{h} G_i$, then $\mathrm{mr}(G) \leq \sum_{i=1}^{h} \mathrm{mr}(G_i)$. A matrix A having $\mathcal{G}(A) = G$ and rank at most $\sum_{i=1}^{h} \mathrm{mr}(G_i)$ can be constructed as the sum of matrices $\alpha_i \tilde{A}_i$, with each $\tilde{A}_i$ obtained by embedding an optimal matrix A_i for G_i in the appropriate place of a $|G| \times |G|$ matrix and the α_i are scalars chosen to avoid the cancellation of off-diagonal entries.*

In a graph, vertices v and w that have the same set of neighbors (except possibly v and w) are called *twins*. Let v and w be twins; if v and w are not adjacent, then v and w are *independent twins*, whereas if v and w are adjacent, then v and w are *clique twins*.

Proposition 3.3. *Let G be a graph with twin vertices v and w.*

- *If v and w are independent twins and $G - w$ has an optimal matrix A such that the diagonal entry $a_{vv} = 0$, then $\mathrm{mr}(G) = \mathrm{mr}(G - w)$.*
- *If v and w are clique twins and $G - w$ has an optimal matrix A such that the diagonal entry $a_{vv} \neq 0$, then $\mathrm{mr}(G) = \mathrm{mr}(G - w)$.*

Proof. In either case, given such an optimal matrix A for $G - w$, a symmetric matrix $\tilde{A}$ such that $\mathcal{G}(\tilde{A}) = G$ can be constructed by duplicating the row and column of A associated with v. Thus $\mathrm{mr}(G - w) \leq \mathrm{mr}(G) \leq \mathrm{rank}(\tilde{A}) = \mathrm{rank}(A) = \mathrm{mr}(G - w)$. □

Many of the matrices used to prove the following proposition appear in [11].

Proposition 3.4. *Let C_n be an n-cycle with the vertices labeled consecutively around the cycle and $n \geq 4$. There exists an optimal matrix of the form $\mathcal{A}(C_n) + D$ with D being a diagonal matrix that has at least one zero diagonal entry and one nonzero diagonal entry.*

Proof. It suffices to exhibit such a matrix for each n.

- For $n \equiv 0 \mod 4$, let $D = \mathrm{diag}(\mathrm{repeat}(0, -1, 0, 1))$.
- [11] For $n \equiv 1 \mod 4$ and $n \geq 13$, let $D = \mathrm{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, \ldots, 0)$.
- [11] For $n \equiv 2 \mod 4$ and $n \geq 10$, let $D = \mathrm{diag}(1, 1, 1, 1, 1, 1, 0, 0, 0, \ldots, 0)$.
- [11] For $n \equiv 3 \mod 4$ and $n \geq 7$, let $D = \mathrm{diag}(1, 1, 1, 0, 0, 0, \ldots, 0)$.
- For $n = 5, 9$, let $D = \mathrm{diag}(-1, -1, -1, 0, \ldots, 0)$.
- For $n = 6$, let $D = \mathrm{diag}(1, 1, 2, 1, 1, 0)$.

Here, 'repeat()' means the sequence enclosed in parentheses appears as many times as needed (possibly zero times) to obtain a vector of the correct length. For $n \equiv 0 \mod 4$, $\mathcal{A}(C_n) + \mathrm{diag}(\mathrm{repeat}(0, -1, 0, 1))$ has the two independent null vectors $[\mathrm{repeat}(-1, 0, 1, 0)]^T$ and $[\mathrm{repeat}(-1, -1, 0, 1)]^T$. The statements for $n = 5, 6, 9$ are readily verified by direct computation. □

Corollary 3.5. *If G is a graph obtained from $C_k, k \geq 4$, by adding one or more independent twins (or one or more clique twins) of a single vertex, then $\mathrm{mr}(G) = k - 2$.*

Proposition 3.6. *For any polygonal path L and any vertex v, there exist optimal matrices A_0, A_1 having zero and nonzero v, v-entries respectively.*

Proof. Note that a polygonal path is a union of cycles, so an optimal matrix can be constructed as the sum of embedded optimal cycle matrices, as in Observation 3.2. Thus the result follows immediately from Proposition 3.4 except in the case that v is a vertex of a triangle and a zero entry is desired. If v is a vertex of degree at least 3, then v is on at least two cycles, and a zero diagonal at v can be created by cancellation. The only remaining case is where v is the degree two vertex of a terminal 3-cycle of L, and we need to obtain a zero entry. It suffices to produce the required matrix for a polygonal path consisting of a 3-cycle adjacent to an $(m - 1)$-cycle, as this can then be embedded and added to an embedding corresponding to the rest of the polygonal path (overlapping on an edge of the $(m - 1)$-cycle).

Let L_m be a polygonal path consisting of a triangle and a cycle, labeled as in Figure 1. We exhibit an $m \times m$ matrix A_m such that $\mathcal{G}(A_m) = L_m$ and $(A_m)_{11} = 0$, and two independent null vectors $\mathbf{v}_m, \mathbf{w}_m$ for A_m.

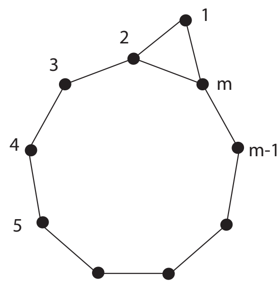

Figure 1: The polygonal path L_m consisting of a 3-cycle and an $(m-1)$-cycle

- For $m \equiv 1 \mod 3, m \geq 4$, let $A = \mathcal{A}(L_m) + \text{diag}(0, 1, \dots, 1, 0, 1)$, $\mathbf{v}_m = [\text{repeat}(-1, 0, 1), 0]^T$, and $\mathbf{w}_m = [0, \text{repeat}(-1, 0, 1)]^T$.
- For $m \equiv 2 \mod 3, m \geq 5$, let $A = \mathcal{A}(L_m) + \text{diag}(0, 1, \dots, 1, -1, -1, 0)$, $\mathbf{v}_m = [\text{repeat}(-1, 0, 1), 1, 0]^T$, and $\mathbf{w}_m = [1, -1, \text{repeat}(-1, 2, -1), -1, 0, 1]^T$.
- For $m \equiv 0 \mod 3, m \geq 6$, let $A = \mathcal{A}(L_m) + \text{diag}(0, 1, \dots, 1, 2, 1, 3)$, $\mathbf{v}_m = [\text{repeat}(-1, 0, 1), -1, 1, 0]^T$, and $\mathbf{w}_m = [-2, \text{repeat}(-1, 2, -1), 0, 1]^T$. □

Corollary 3.7. *If G is a graph obtained from a polygonal path L by adding one or more independent twins (or one or more clique twins) of a single vertex, then* $\text{mr}(G) = |L| - 2$.

4 Computation of minimum ranks of graphs of order 7

The program was run on all graphs of order at most 7, and the minimum ranks of 1210 of 1252 (nonisomorphic) graphs were computed. This includes all 208 graphs of order at most 6 (whose minimum rank was known but not used when running the program to determine the minimum rank of these graphs) and 1002 of the 1044 graphs of order 7. The results in Section 3 and additional methods were used as described below to determine the minimum ranks of the remaining 48 graphs of order 7. The graph numbers below (e.g., G558) are taken from [14]. The minimum ranks of all graphs of order at most 7 are available in the spreadsheet [6]. The published program [7] now has access to the minimum ranks of all graphs at most order 7, using the results that we have obtained.

Proposition 4.1.

1. $\text{mr}(G679) = 4$ *by lower bounds found by the program and Theorem 3.1.*
2. $\text{mr}(G558) = 3$ *by a lower bound found by the program and Corollary 3.5.*
3. $\text{mr}(G669) = \text{mr}(G678) = \text{mr}(G791) = 3$ *by lower bounds found by the program and Corollary 3.7.*
4. $\text{mr}(G1086) = \text{mr}(G1135) = 3$ *by lower bounds found by the program and by using Observation 3.2 (and the given subgraphs) to guarantee the existence of a matrix of rank 3: G1086* $(K_4 \cup G50)$*, G1135* $(G204 \cup K_3)$.

Proposition 4.2. *The minimum rank of each of the following graphs is 3. In each case the program found a lower bound of 3 and we constructed a matrix of rank 3: G721, G801, G812, G831, G832, G846, G863, G873, G878, G913, G918, G924, G932, G944, G953, G956, G958,* *G970, G990, G995, G996, G1002, G1005, G1028, G1060, G1075, G1077, G1087, G1095, G1099, G1104, G1146, G1167, G1205, G1212.*

The matrices referred to in Proposition 4.2 are available for examination on the AIM workshop webpage [2] and on ArXiv [6] in both PDF and *Sage* source code.

5 Appendix: Method used by the program

The minimum rank program is available on the AIM website [2] and on ArXiv [7] in both PDF and *Sage* source code. The program contains documentation that explains how to use the program in *Sage*, provides examples of the use of the program to compute the minimum ranks of various graphs, and describes the options to provide additional output. The default version of the program determines all possible information and checks for consistency; the program also allows the user to specify that fewer tests be performed. Additionally, the program contains a function to export data that was used to generate the table in [6].

The default version of the program uses the following general method:

1. Separate the graph into its connected components and work on each component separately.
2. If the order is less than eight, look up the minimum rank. The published version of the program [7] uses the data for order at most seven that is described in Section 4.
3. Determine the lower and upper bounds on minimum rank given by Theorems 2.2 and 2.3.
4. If G is a tree, determine the exact bound given by Theorem 2.1.
5. Search for a cut-vertex and apply Theorem 2.4 and recursion.

Note: Finding the clique cover number is NP-hard [10]. An additional function to compute an approximate bound more efficiently is also included. The zero forcing number is found by brute force. In our experience, the program can almost always compute minimum rank bounds on a graph of order at most 10 in a few seconds, and on a graph of order 15 in less than ten minutes, on a 1.67 GHz Mac Powerbook G4.

Contact the authors for further information.

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