ON THE GRAPH COMPLEMENT CONJECTURE FOR MINIMUM RANK*

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Abstract. The minimum rank of a graph has been an interesting and well studied parameter investigated by many researchers over the past decade or so. One of the many unresolved questions on this topic is the so-called graph complement conjecture, which grew out of a workshop in 2006. This conjecture asks for an upper bound on the sum of the minimum rank of a graph and the minimum rank of its complement, and may be classified as a Nordhaus-Gaddum type problem involving the graph parameter minimum rank. The conjectured bound is the order of the graph plus two. Other variants of the graph complement conjecture are introduced here for the minimum semidefinite rank and the Colin de Verdière type parameter ν. We show that if the ν-graph complement conjecture is true for two graphs then it is true for the join of these graphs. Related results for the graph complement conjecture (and the positive semidefinite version) for joins of graphs are also established. We also report on the use of recent results on partial k-trees to establish the graph complement conjecture for graphs of low minimum rank.

Key words. Minimum rank, Nordhaus-Gaddum type, graph, graph complement, maximum multiplicity, join of graphs, Colin de Verdière type parameters, minimum semidefinite rank, k-trees, Hadwiger number.

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1. Introduction. All matrices discussed are real and symmetric; the set of $n \times n$ real symmetric matrices will be denoted by $S_n(\mathbb{R})$. A *graph* $G = (V, E)$ means a simple undirected graph (no loops, no multiple edges) with a nonempty set of vertices V and edge set E (an edge is a two-element subset of vertices). Recall that the *complement of a graph* $G = (V, E)$ is the graph $\overline{G}$ on the same set of vertices, and with edges $\{i, j\}$, $i \neq j$ exactly when $\{i, j\} \notin E$. The *order of* G, denoted by $|G|$, is simply the cardinality of its vertex set V.

Studying collections of matrices associated to a combinatorial object, such as a graph, has long been a topic of interest to both the linear algebra community and to the combinatorial community. One instance of this general study is the so-called *minimum rank problem for graphs.* In general, the minimum rank problem for a graph G asks to determine the smallest possible rank over the collection of all real symmetric matrices $A = [a_{ij}]$ with the adjacency property that for each $i \neq j$, $a_{ij} \neq 0$ if and only if $\{i, j\}$ is an edge in G. In general, for $A \in S_n(\mathbb{R})$, the *graph* of A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : \ a_{ij} \neq 0, 1 \leq i < j \leq n\}$.

Let G be a graph. The *set of symmetric matrices described by* G is defined to be

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\},$$

and the quantity we study is the *minimum rank of* G, defined and denoted by

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}.$$

Since the main diagonal of any member in $\mathcal{S}(G)$ is ignored in determining $\mathcal{G}(A)$, it is clear that $0 \leq \mathrm{mr}(G) \leq n - 1$; and if G is nontrivial we have $1 \leq \mathrm{mr}(G) \leq n - 1$. A basic example along these lines is the path on n vertices, which is known to be the only graph (on n vertices) with minimum rank equal to $n - 1$ (see [13]).

The general and important matter of resolving the minimum rank of an arbitrary graph is a very difficult and open problem. However, considerable research on this issue has lead to significant progress on many facets of it. For example, general formulas are known for the minimum rank of trees and unicyclic graphs, and complete descriptions of the graphs G for which $\mathrm{mr}(G) = 1, 2, n - 2, n - 1$ have been recorded in the literature (see, for example, [12] and the references therein).

The topic of minimum rank of graphs has garnered sufficient attention in the literature to a point where it became a core subject at an American Institute of Mathematics workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns [2]. A result of this workshop was a number of suggested problems one of which has become known as the "Graph Complement Conjecture" or GCC for short. The GCC can be stated as the following conjecture about the minimum rank of G and its complement.

Conjecture 1.1 (GCC Conjecture). *For any graph G,*

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2.$$

For example, if $G = C_5$, the cycle on 5 vertices, then $\mathrm{mr}(C_5) = 3$ and $\mathrm{mr}(\overline{C_5}) = \mathrm{mr}(C_5) = 3$. Hence, $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) = 3 + 3 < 5 + 2$.

It is worth noting that the actual question posed at this 2006 AIM workshop was: How large can $\mathrm{mr}(G) + \mathrm{mr}(\overline{G})$ be? From this two possibilities arise (see also [8]):

Question 1) Does there exist a constant $c \geq 2$ such that $\mathrm{mr}(G)+\mathrm{mr}(\overline{G}) \leq |G|+c$? If so, what is the smallest such c?

Question 2) Find the smallest constant $d \leq 2$ such that $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq d|G|$.

The condition $c \geq 2$ in Question 1 follows from examination of the path on 4 vertices, written as P_4. Observe that $\mathrm{mr}(P_4) + \mathrm{mr}(\overline{P_4}) = 6 = 4 + 2$, which implies $c \geq 2$. On the other hand, since $\mathrm{mr}(G) \leq |G| - 1$ for any graph it follows that d (in Question 2) can be chosen to be at most 2. It has been suspected for some time that $c = 2$ is the correct bound for Question 1, hence the GCC.

Since the original GCC was recorded, further analysis has lead to stronger conjectures on the minimum rank of all positive semidefinite matrices in $\mathcal{S}(G)$. We let $\mathrm{mr}_+(G)$ denote the minimum rank of all matrices A in $\mathcal{S}(G)$ with the additional constraint that A be positive semidefinite.

Conjecture 1.2 (GCC_+ Conjecture). *For any graph G,*

$$\mathrm{mr}_+(G) + \mathrm{mr}_+(\overline{G}) \leq |G| + 2.$$

It is clear that GCC_+ represents a stronger inequality than does GCC, and thus the bound of $|G| + 2$ is best possible for GCC_+ (note that for GCC_+, any tree T that contains an induced P_4 has equality in the bound, because $\mathrm{mr}_+(T) = |T| - 1$ and $\mathrm{mr}_+(\overline{T}) = 3$ (see [3]).

We note here that both GCC and GCC_+ (and later GCC_ν) fall into the class of so-called Nordhaus-Gaddum type problems (see [1], for example) in that they involve bounding the sum of a graph parameter evaluated at a graph G and its complement $\overline{G}$. Nordhaus-Gaddum type problems have been studied for many different graph parameters, including chromatic number, independence number, domination number and others such as the Hadwiger number (see [20]). Since the matrix community has been referring to these suspected inequalities as graph complement conjectures, we continue to use these names within this work as well.

Observe that if we define the *maximum nullity of G* as

$$\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\},$$

and the *maximum positive semidefinite nullity of G* as

$$\mathrm{M}_+(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G), A \text{ is positive semidefinite}\},$$

Conjectures 1.1 and 1.2 are equivalent to

$$\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq |G| - 2, \tag{1.1}$$

$$\mathrm{M}_+(G) + \mathrm{M}_+(\overline{G}) \geq |G| - 2. \tag{1.2}$$

A related conjecture (see Conjecture 1.7 below) was made in [21], using the Colin de Verdière number $\mu(G)$ that is equal to the maximum nullity among all matrices satisfying several conditions including the Strong Arnold Hypothesis (see definitions below). The parameter μ, which is used to characterize planarity, is the first of several parameters that require the Strong Arnold Hypothesis and bound the maximum nullity from below (called *Colin de Verdière type parameters*). A real symmetric matrix A satisfies the *Strong Arnold Hypothesis* provided there does not exist a nonzero real symmetric matrix X satisfying $AX = 0$, $A \circ X = 0$, and $I \circ X = 0$, where $\circ$ denotes the Hadamard (entry-wise) product and I is the identity matrix. The Strong Arnold Hypothesis is equivalent to the requirement that certain manifolds intersect transversally (see [18]). The parameter $\mu(G)$ is defined ([9] in English) to be the maximum nullity among symmetric matrices $A = [a_{ij}] \in \mathcal{S}(G)$ that satisfy:

- A satisfies the Strong Arnold Hypothesis.
- For all $i \neq j$, $a_{ij} \leq 0$.
- A has exactly one negative eigenvalue (counting multiplicity).

In [10] Colin de Verdière introduced the parameter $\nu(G)$, defined to be the maximum nullity among positive semidefinite matrices $A \in \mathcal{S}(G)$ that satisfy the Strong Arnold Hypothesis. Evidently, for every graph G, $\nu(G) \leq \mathrm{M}_+(G) \leq \mathrm{M}(G)$. So it is natural to ask whether GCC_+ can be extended to ν:

Conjecture 1.3 (GCC_ν Conjecture). *For any graph G,*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2, \tag{1.3}$$

Thus (1.3) is stronger than GCC_+ (and hence GCC) in general (cf. (1.1) and (1.2)). Since some of the arguments later are done in terms of rank, it is instructive to associate a name to the rank parameter associated with the nullity parameter ν.

Definition 1.4. For a graph G, define $\mathrm{mr}_\nu(G) = |G| - \nu(G)$.

With this definition, Conjecture 1.3 becomes

$$\mathrm{mr}_\nu(G) + \mathrm{mr}_\nu(\overline{G}) \leq |G| + 2. \tag{1.4}$$

An important property of Colin de Verdière-type parameters is minor monotonicity. The *contraction* of edge $e = \{u, v\}$ of G is obtained by identifying the vertices u and v, deleting any loops that arise in this process, and replacing any multiple edges by a single edge. A *minor* of G arises by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges. A graph parameter β is *minor monotone* if for any minor H of G, $\beta(H) \leq \beta(G)$ and $\beta(G) = \beta(H)$ if G is isomorphic to H. In [9] and [10] it is shown that μ and ν are minor monotone.

For any graph G, the *Hadwiger number* $\mathrm{h}(G)$ is the maximum size of a clique minor in G. It is straightforward to verify that $\nu(K_s) = s - 1$ whenever $s > 1$, so by minor monotonicity we have:

Observation 1.5. *Let G be a graph.*

$$\mathrm{M}(G) \geq \mathrm{M}_+(G) \geq \nu(G) \geq \mathrm{h}(G) - 1.$$

Given this relationship (and the fact that it is common to use $\mathrm{h}(G) - 1$ as a lower bound when establishing the value of $\nu(G)$, it is reasonable to ask whether a version of GCC is true for the Hadwiger number. The bound would be $(\mathrm{h}(G)-1)+(\mathrm{h}(\overline{G})-1) \geq |G| - 2$ or equivalently,

$$\mathrm{h}(G) + \mathrm{h}(\overline{G}) \geq |G|. \tag{1.5}$$

There is a body of literature on Hadwiger number Nordhaus-Gaddum type problems, and it is known [20] that (1.5) is not true for all graphs G (or even for most graphs of large order). The next example gives a specific graph for which (1.5) fails. As is standard, we let $\kappa(G)$ denote the *vertex connectivity of* G, i.e., if G is not complete, it is the smallest number k such that there is a set of vertices S, with $|S| = k$, for which $G - S$ is disconnected (by convention, $\kappa(K_n) = n - 1$). As noted in [17], as a consequence of results in [22] and [23], for every graph G, $\kappa(G) \leq \nu(G)$.

Example 1.6. Let G_{12} be the icosahedral graph, which has order 12, is 5-regular, and is planar. Thus G_{12} cannot have a K_5 minor. So in order for G_{12} to satisfy (1.5),

$\overline{G_{12}}$ would need to have a K_8 minor. This is impossible, since for any minor that has 8 vertices, we must partition the 12 vertices of $\overline{G_{12}}$ into 8 sets (associated with the 8 vertices of the minor), requiring that there be a set with only one vertex of $\overline{G_{12}}$, hence a vertex of degree at most 6 in the minor, because $\overline{G_{12}}$ is 6-regular.

Note that $\kappa(G_{12}) = 5$ and $\kappa(\overline{G_{12}}) = 6$, so $\nu(G_{12}) + \nu(\overline{G_{12}}) \geq 11 > |G_{12}| - 2$. So G_{12} satisfies GCC_ν and hence GCC and GCC_+.

In 1997 the following related conjecture was made:

Conjecture 1.7 (GCC_μ Conjecture). [21, p. 512][1] *For any graph G,*

$$\mu(G) + \mu(\overline{G}) \geq |G| - 2. \tag{1.6}$$

It is a consequence of results in [21] and [9] that GCC_μ holds for all planar graphs, so the icosahedral graph in Example 1.6 does satisfy GCC_μ. Since, in general, μ is not comparable to ν or M_+, it follows that GCC_μ does not imply either GCC_ν or GCC_+, but it does imply GCC.

In Section 2 we turn to the case of k-trees, and making use of some recent analysis on the minimum ranks of the complements of k-trees, we will establish that the GCC, GCC_+, and GCC_ν are valid for this class of graphs as well. In Section 3 we consider joins of graphs. The first subsection involves GCC_ν and we prove, via induction, that if two graphs satisfy GCC_ν so will their join. In the next subsection we will verify that if a modified version of GCC (or GCC_+) holds for two graphs, then GCC (or GCC_+) holds for their join.

2. k-trees and the Graph Complement Conjecture. A graph G is called a *k-tree* if it can be constructed inductively by starting with K_{k+1} and connecting each new vertex to the vertices of an existing K_k (i.e., a k-clique). Every clique in a k-tree is part of a $(k+1)$-clique, and a k-tree is a k-connected chordal graph with maximum clique size $k+1$. The graph depicted in Figure 2.1, known as the supertriangle, is an example of a 2-tree on 6 vertices. A graph G is called a *partial k-tree* if G is a subgraph of a k-tree. Observe that each graph is a partial k-tree for some value of k (for example, $k = |G| - 1$ always works). The minimum k for which G is a partial k-tree is equal to the tree-width $\mathrm{tw}(G)$ of G (see, for example, [7, F12 p. 111]).

The main purpose of this section is to verify that the graph complement conjecture (and its variants GCC_+ and GCC_ν) hold for k-trees, for certain classes of partial k-trees, and for small graphs. Much of the following analysis relies on recent work by H. van der Holst and J. Sinkovic (see [19]), which we state here for completeness.

[1]The reader is warned that in [21] the notation $\nu(G)$ means something entirely different from the Colin de Verdière parameter $\nu(G)$ used in this paper.

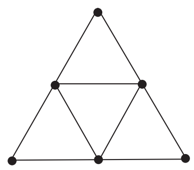

FIG. 2.1. *The supertriangle* T_3

THEOREM 2.1. [19] *If* G *is a partial* k*-tree, then*

$$\nu(\overline{G}) \geq |G| - k - 2.$$

THEOREM 2.2. [19] *If* G *is a partial 3-tree, then*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

Finally, in the same work they observe (using the above results and results from [22], [23]) that GCC_ν holds for k-connected partial k-trees.

COROLLARY 2.3. [19] *If* G *is a* k*-connected partial* k*-tree, then*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

In particular, if G *is a* k*-tree, then* G *satisfies* GCC_ν*.*

As a consequence of Theorem 2.1 and the fact that $\mathrm{tw}(G)$ is the minimum k such that G is a partial k-tree, we have the following corollary.

COROLLARY 2.4. *If* GCC_ν *fails for a graph* G*, then* $\kappa(G) < \mathrm{tw}(G)$*.*

Since the Hadwiger number minus one is a lower bound for ν, we have the following consequence of Theorem 2.1.

COROLLARY 2.5. *If* G *is a partial* k*-tree with* $\mathrm{h}(G) = k + 1$*, then* G *satisfies*

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

Proof. Apply Observation 1.5 and Theorem 2.1. ∎

OBSERVATION 2.6. *If* G *is a graph for which* $\nu(\overline{G}) \geq |G| - \mathrm{h}(G) - 1$ *(respectively,* $\mathrm{mr}_+(\overline{G}) \leq \mathrm{h}(G) + 1$*,* $\mathrm{mr}(\overline{G}) \leq \mathrm{h}(G) + 1$*) then* G *will satisfy* GCC_ν *(respectively,* GCC_+*,* GCC*).*

Previously, GCC was known to hold for all graphs on seven or fewer vertices, since for all such graphs, the minimum ranks have been exhaustively computed [11]. Here we extend this result (and eliminate the need for exhaustive computation) and determine properties of a minimum counterexample to GCC, GCC_+, or GCC_ν.

COROLLARY 2.7. *If* GCC_ν *fails for some graph* G*, then* $\nu(G) \geq 3$ *and* $\nu(\overline{G}) \geq 3$.

Proof. By Theorem 2.2, neither G nor $\overline{G}$ can be a partial 2-tree. Since a graph is not a partial 2-tree if and only if it has a K_4 minor [7, F31, p. 112], $\mathrm{h}(G), \mathrm{h}(\overline{G}) \geq 4$ and $\nu(G), \nu(\overline{G}) \geq 3$. □

COROLLARY 2.8. *If* G *is a graph with* $|G| \leq 8$*, then* GCC_ν *holds for* G.

Proof. If GCC_ν fails, then by Corollary 2.7 we have

$$|G| - 2 > \nu(G) + \nu(\overline{G}) \geq 3 + 3.$$

This reduces to $|G| > 8$, as desired. □

Note that GCC_μ (Conjecture 1.7) holds for any graph G of order at most 7, since for such a graph either G or $\overline{G}$ must be planar, and, as observed in the paragraph following Conjecture 1.7, GCC_μ holds for all planar graphs (see also [21]).

Since it is established that any graph having tree-width at most three satisfies GCC_ν (and hence GCC), we can improve the bounds in Corollaries 2.7 and 2.8 for GCC by examining the forbidden minors for tree-width three, which are K_5, the complete tripartite graph $K_{2,2,2}$, the graph V_8 shown in Figure 2.2 with the numbering that will be used throughout the discussion of this graph, and the Cartesian product $C_5 \,\square\, P_2$ (see [3] for the definition of Cartesian product). We use the minor monotone Colin de Verdière-type parameter ξ, introduced in [5] and defined to be the maximum nullity over all matrices A in $\mathcal{S}(G)$ that satisfy the Strong Arnold Hypothesis. Clearly $\nu(G) \leq \xi(G) \leq \mathrm{M}(G)$ for all G.

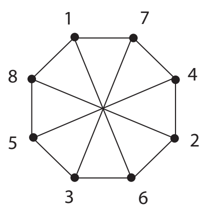

FIG. 2.2. *The graph* V_8

PROPOSITION 2.9. $\nu(K_{2,2,2}) = \xi(K_{2,2,2}) = 4$, $\xi(V_8) = 4$, *and* $\xi(C_5 \,\square\, P_2) = 4$.

Proof. For $\nu(K_{2,2,2}) = \xi(K_{2,2,2}) = 4$, note that $\mathrm{mr}(K_{2,2,2}) = 2$ and let

$$B = \begin{bmatrix} 1 & 1 & 2 & 1 & -2 & 1 \\ 1 & 2 & 1 & -1 & 1 & -2 \end{bmatrix}.$$

Then the positive semidefinite matrix $A = B^T B \in \mathcal{S}(K_{2,2,2})$, $\operatorname{rank} A = 2$, and it is straightforward to verify that A satisfies the Strong Arnold Hypothesis (which can be checked using a computer symbolic package).

For $\xi(V_8) = 4$, note that $\mathrm{M}(V_8) = 4$ (see the Möbius ladder in [3]) and let

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & -2 & -2 & -2 \\ 0 & 1 & 0 & -2 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & -2 & -1 & 0 \\ 0 & -2 & 0 & 2 & -2 & 0 & -2 & 0 \\ 0 & 0 & -2 & -2 & 2 & 0 & 0 & -2 \\ -2 & -2 & -2 & 0 & 0 & 2 & 0 & 0 \\ -2 & 0 & -1 & -2 & 0 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}(V_8).$$

Since $\operatorname{rank} A = 4$ and A satisfies the Strong Arnold Hypothesis, $4 \leq \xi(V_8) \leq \mathrm{M}(V_8) = 4$.

For $\xi(C_5 \,\square\, P_2) = 4$, note that $\mathrm{M}(C_5 \,\square\, P_2) = 4$ [3] and let

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Then $A \in \mathcal{S}(C_5 \,\square\, P_2)$, $\mathrm{null}(A) = 4$, and A satisfies the Strong Arnold Hypothesis. ∎

COROLLARY 2.10. *If* GCC *fails for some graph G, then* $\mathrm{mr}(G) \leq |G| - 4$ *and* $\mathrm{mr}(\overline{G}) \leq |G| - 4$.

Proof. By Theorem 2.2, neither G nor $\overline{G}$ can be a partial 3-tree. A graph is not a partial 3-tree if and only if it has at least one of the graphs K_5, $K_{2,2,2}$, V_8, $C_5 \,\square\, P_2$ as a minor [7, F33, p. 112]. Thus by Proposition 2.9, G has $4 \leq \xi(G) \leq \mathrm{M}(G)$. Thus $\mathrm{mr}(G) \leq |G| - 4$, and similarly for $\overline{G}$. ∎

Corollary 2.11. *If G is a graph with $|G| \leq 10$, then* GCC *holds for G.*

The method used to establish Corollaries 2.10 and 2.11 does not work for GCC_ν or GCC_+, since $\mathrm{mr}_+(V_8) = 5$. To see this, we attempt to construct a vector representation in $\mathbb{R}^4$ of the vertices of V_8 (as labeled in Figure 2.2). Without loss of generality the first three vectors (representing vertices 1, 2, and 3) are the first three standard basis vectors. Then vector 4 is orthogonal to 1 and 3, but not to 2 and not a multiple of 2, so it is a multiple of $(0, 1, 0, a)$ for a nonzero a; similarly 5 is $(0, 0, 1, b)$ with $b \neq 0$. Then it follows that vector 6 must be a multiple of $(c, a, b, -1)$, for c nonzero. Finally, vector 7 is in the null space of columns 2, 5, and 6, so it is a multiple of $(1 + b^2, 0, -bc, c)$, and vector 8 is a multiple of $(1 + a^2, -ac, 0, c)$. But vectors 7 and 8 are required to be orthogonal, implying $1 + a^2 + b^2 + a^2b^2 + c^2 = 0$, a contradiction.

3. Joins of Graphs.

All unions and joins in this paper involve disjoint graphs. Recall that, if G_1 and G_2 are disjoint graphs, the *union* and the *join* of G_1 and G_2, denoted respectively by $G_1 \cup G_2$ and $G_1 \vee G_2$, are the graphs defined by

$$\begin{aligned} V(G_1 \cup G_2) = V(G_1 \vee G_2) \ &= \ V(G_1) \cup V(G_2); \\ E(G_1 \cup G_2) = E(G_1) \cup E(G_2);& \\ E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup E,& \end{aligned}$$

where E consists of all the edges $\{u, v\}$ with $u \in V(G_1)$, $v \in V(G_2)$. A union or a join of r graphs is defined inductively by

$$\bigcup_{i=1}^{r} G_i = \left(\bigcup_{i=1}^{r-1} G_i \right) \cup G_r, \qquad \bigvee_{i=1}^{r} G_i = \left(\bigvee_{i=1}^{r-1} G_i \right) \vee G_r.$$

Some of the results in this section rely on a "Rotation Lemma" as it was referred to in [4, Lemma 2.3] that pertains to the construction of certain types of isometries in an indefinite inner product space. To this end, we require some additional notation regarding the inertia of a symmetric matrix. For any $n \times n$ symmetric matrix A, we define the *inertia* of A as the triple $(i_+(A), i_-(A), i_0(A))$, consisting of the number of positive, negative, and zero eigenvalues (counting multiplicity) of A, respectively. Clearly, $i_0(A) = n - i_+(A) - i_-(A)$, and A is positive semidefinite if and only if $i_-(A) = 0$.

Definition 3.1. Suppose is an $n \times n$ A symmetric matrix. A *nonzero* (h, k)-*representation of* A is a $(h + k) \times n$ matrix

$$\begin{bmatrix} P_A \\ N_A \end{bmatrix}$$

with no zero columns such that P_A has h rows, N_A has k rows and $A = P_A^T P_A - N_A^T N_A$.

Observe that for such a representation to exist, we must have that $h \geq i_+(A)$ and $k \geq i_-(A)$. In fact, the matrix P_A represents the positive inertia of A, and N_A represents the negative inertia of A. Also note that if A is positive semidefinite, then N_A may be chosen to be the zero matrix.

Any symmetric matrix having all columns nonzero has a nonzero (h,k)-representation whenever both $h \geq i_+(A)$ and $k \geq i_-(A)$. However, not every symmetric matrix has a nonzero $(i_+(A), i_-(A))$-representation, due to the presence of zero columns. In particular if G is a graph with no isolated vertices, then any matrix $A \in \mathcal{S}(G)$ with $\operatorname{rank} A = \operatorname{mr}(G)$ has a nonzero (h,k)-representation with $h + k = \operatorname{mr}(G)$. Finally, observe that if A has a nonzero (h,k)-representation, then, by padding both P_A and N_A with zero rows as needed, it follows that A has a nonzero (h', k')-representation for $h' \geq h$ and $k' \geq k$.

A matrix Q of order $h+k$ is said to be (h,k)-*orthogonal* if $Q^T \tilde{I} Q = \tilde{I}$, where

$$\tilde{I} = \begin{bmatrix} I_h & 0 \\ 0 & -I_k \end{bmatrix}$$

(I_s refers to the $s \times s$ identity matrix). Given a nonzero (h,k)-representation $\begin{bmatrix} P_A \\ N_A \end{bmatrix}$ for A and a (h,k)-orthogonal matrix Q, it follows that $Q \begin{bmatrix} P_A \\ N_A \end{bmatrix}$ is also a nonzero (h,k)-representation for A. The previous fact can be verified by direct computation.

We are now in a position to state a revised version of the rotation lemma that was presented in [4, Lemma 2.3]. We remark here that the proof is basically the same as the one presented in [4] and is not repeated here.

Lemma 3.2. *Let G and H be two graphs and let $A \in \mathcal{S}(G)$ and $B \in \mathcal{S}(H)$. Suppose A and B each have nonzero (h,k)-representations $\begin{bmatrix} P_A \\ N_A \end{bmatrix}$ and $\begin{bmatrix} P_B \\ N_B \end{bmatrix}$, respectively with $h \geq 2$. Then there exists an (h,k)-orthogonal matrix Q such that*

$$\begin{bmatrix} P_A & P'_B \\ N_A & N'_B \end{bmatrix}$$

is a nonzero (h,k)-representation of a matrix in $\mathcal{S}(G \vee H)$ with

$$\begin{bmatrix} P'_B \\ N'_B \end{bmatrix} = Q \begin{bmatrix} P_B \\ N_B \end{bmatrix}$$

Note that in Lemma 3.2 we must have

$$h \geq \max\{i_+(A), i_+(B)\} \text{ and } k \geq \max\{i_-(A), i_-(B)\}.$$

Also observe that if $k = 0$, we obtain a result for positive semidefinite matrices in $\mathcal{S}(G)$ and $\mathcal{S}(H)$

In Section 3.1 we prove that if G and H are graphs each satisfying GCC_ν then $G \vee H$ (or equivalently, $G \cup H$) satisfies GCC_ν. Related results for GCC and GCC_+, which are substantially more complicated, are proved in Section 3.2.

3.1. GCC_ν **for joins of graphs.** The Colin de Verdière type parameters have the important property that instead of summing over connected components (like maximum nullity or minimum rank), they take the maximum.

THEOREM 3.3. [10] *For disjoint graphs G and H, $\nu(G \cup H) = \max\{\nu(G), \nu(H)\}$, so*

$$\mathrm{mr}_\nu(G \cup H) = |G| + |H| - \max\{\nu(G), \nu(H)\}.$$

For example, whereas $\mathrm{mr}(G) = 1$ implies $G = K_r \cup \overline{K_s}$, $\mathrm{mr}_\nu(G) = 1$ implies $G = K_r$, $r \geq 2$ or $G = K_1 \cup K_1$.

DEFINITION 3.4. A *ν-optimal* matrix for a graph G is a positive semidefinite matrix $A \in \mathcal{S}(G)$ that satisfies the Strong Arnold Hypothesis and has $\mathrm{null}\, A = \nu(G)$ (or equivalently, $\mathrm{rank}\, A = \mathrm{mr}_\nu(G)$).

LEMMA 3.5. *If G has an edge then there exists a ν-optimal matrix A for G such that every column of A has a nonzero entry.*

Proof. For any $B \in \mathcal{S}(G)$, B is a block diagonal matrix with diagonal blocks associated with the connected components of G. If there is only one component, the result is immediate. If $B \in \mathcal{S}(G)$ satisfies the Strong Arnold Hypothesis, then at most one of the diagonal blocks of B is singular [5, Lemma 3.1]. A ν-optimal matrix must have the singular block associated with a component having maximum value of ν. Since $\nu(K_1) = 1$, we can choose to have the singular block associated with a component that has an edge. □

LEMMA 3.6. *Suppose H is an induced subgraph of G. Then* $\mathrm{mr}_\nu(H) \leq \mathrm{mr}_\nu(G)$.

Proof. Suppose A is a ν-optimal matrix for G, and let B be the principal submatrix of A that corresponds to the induced subgraph H. By renumbering if necessary we may assume $A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$. From properties of positive semidefinite matrices, it follows that B is a positive semidefinite matrix with graph H, so once we show that

B satisfies the Strong Arnold Hypothesis,

$$\mathrm{mr}_\nu(H) \leq \operatorname{rank} B \leq \operatorname{rank} A = \mathrm{mr}_\nu(G).$$

Note that the column inclusion property of positive semidefinite matrices guarantees that there exists a matrix E such that $C = BE$. So if Y is a symmetric matrix such that $BY = 0, B \circ Y = 0$ and $I \circ Y = 0$, define $X = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$AX = \begin{bmatrix} BY & 0 \\ C^T Y & 0 \end{bmatrix} = \begin{bmatrix} BY & 0 \\ E^T BY & 0 \end{bmatrix} = 0.$$

Since A satisfies the Strong Arnold Hypothesis, $X = 0$, so $Y = 0$ and B satisfies the Strong Arnold Hypothesis. □

THEOREM 3.7. *Let G and H be graphs. If*

1. *G and H each have an edge, or*
2. *either G has an edge and $H = \overline{K_r}$, and $\mathrm{mr}_\nu(G) \geq r$;*
   or the same is true with the roles of G and H reversed,

then

$$\mathrm{mr}_\nu(G \vee H) = \max\{\mathrm{mr}_\nu(G), \mathrm{mr}_\nu(H)\}.$$

Otherwise,

$$\mathrm{mr}_\nu(G \vee H) = \max\{\mathrm{mr}_\nu(G), \mathrm{mr}_\nu(H)\} + 1.$$

Proof. Assume first one of conditions (1) and (2) is true. In case (1) without loss of generality $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_\nu(H)$. In case (2) without loss of generality G has an edge, $H = \overline{K_r}$, and $\mathrm{mr}_\nu(G) \geq r$, so $\mathrm{mr}_\nu(G) > r - 1 = \mathrm{mr}_\nu(H)$. In either case, $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_\nu(H)$.

Assume first that $\mathrm{mr}_\nu(G) = 1$. Since G has an edge, the case $G = K_1 \cup K_1$ is excluded and $G = K_t$ for some $t \geq 2$. Since $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_\nu(H)$, $\mathrm{mr}_\nu(H) \leq 1$. Furthermore, either H has an edge, in which case $H = K_s$, or $H = K_1$ (because in this case $1 = \mathrm{mr}_\nu(G) \geq |H|$), so $H = K_s$ for some $s \geq 1$. Thus $G \vee H = K_{t+s}$ and $\mathrm{mr}_\nu(G \vee H) = 1 = \max\{\mathrm{mr}_\nu(G), \mathrm{mr}_\nu(H)\}$.

So assume $\mathrm{mr}_\nu(G) \geq 2$. Since G has an edge, by Lemma 3.5 we can choose a ν-optimal matrix A for G such that every column of A has a nonzero entry. If $H \neq \overline{K_r}$, then we can also choose a ν-optimal matrix B for H such that every column of B has a nonzero entry. If $H = \overline{K_r}$, then $r \leq \mathrm{mr}_\nu(G)$ by hypothesis, so we can choose a diagonal matrix $B \in \mathcal{S}(H)$ having all diagonal entries positive and $\operatorname{rank} B = r \le \operatorname{mr}_\nu(G) = \operatorname{rank} A$. Note that $i_+(A) = \operatorname{rank} A = \operatorname{mr}_\nu(G)$. Then, by Lemma 3.2, we may construct a positive semidefinite matrix $C = \begin{bmatrix} A & *^T \\ * & B \end{bmatrix}$ (where $*$ denotes a matrix all of whose entries are nonzero) with $\operatorname{rank} C = i_+(A) = \operatorname{mr}_\nu(G)$. Since A and B satisfy the Strong Arnold Hypothesis, any such matrix C satisfies the Strong Arnold Hypothesis. Thus it follows that

$$\operatorname{mr}_\nu(G \vee H) = \operatorname{mr}_\nu(G) = \max\{\operatorname{mr}_\nu(G), \operatorname{mr}_\nu(H)\},$$

by also applying Lemma 3.6. This completes the proof for the case in which G and H satisfy condition (1) or (2).

For all remaining cases, we may assume that $H = \overline{K_r}$ and $r > \operatorname{mr}_\nu(G)$. Then

$$\operatorname{mr}_\nu(G \vee H) \ge r = \max\{\operatorname{mr}_\nu(G), \operatorname{mr}_\nu(H)\} + 1$$

because if $C \in \mathcal{S}(G \vee H)$ is positive semidefinite, then C contains an $r \times r$ diagonal matrix with positive diagonal (associated with H).

Either G has an edge or $G = \overline{K_s}$ with $s \le r$. If G has an edge, choose a ν-optimal matrix A for G such that every column of A has a nonzero entry. If $G = \overline{K_s}$, chose a positive definite diagonal matrix $A \in \mathcal{S}(G)$. Then choosing a positive definite diagonal matrix $B \in \mathcal{S}(H)$ and arguing as above shows $\operatorname{mr}_\nu(G \vee H) \le r$. $\square$

THEOREM 3.8. *If G and H are graphs that satisfy* GCC_ν*, then $G \vee H$ and $G \cup H$ satisfy* GCC_ν.

Proof. It suffices to prove the result for $G \vee H$. First assume $H = \overline{K_r}$ and $r > \operatorname{mr}_\nu(G)$, so by Theorem 3.7, $\operatorname{mr}_\nu(G \vee H) = r = |H|$. By Theorem 3.3,

$$\begin{aligned} \operatorname{mr}_\nu(\overline{G} \cup K_r) &= |G| + r - \max\{\nu(\overline{G}), r - 1\} \\ &\le |G| + r - (r - 1) \\ &= |G| + 1. \end{aligned}$$

Thus

$$\operatorname{mr}_\nu(G \vee H) + \operatorname{mr}_\nu(\overline{G} \cup \overline{H}) \le |H| + |G| + 1,$$

and $G \vee H$ satisfies GCC_ν.

Now assume G and H satisfy GCC_ν and satisfy condition (1) or (2) of Theorem 3.7, so

$$\operatorname{mr}_\nu(G \vee H) + \operatorname{mr}_\nu(\overline{G} \cup \overline{H}) = \max\{\operatorname{mr}_\nu(G), \operatorname{mr}_\nu(H)\} + |G| + |H| - \max\{\nu(\overline{G}), \nu(\overline{H})\}.$$

Without loss of generality, $\mathrm{mr}_\nu(G) \geq \mathrm{mr}_\nu(H)$.

Suppose first that $\nu(\overline{G}) \geq \nu(\overline{H})$. Then

$$|G| + |H| - \max\{\nu(\overline{G}), \nu(\overline{H})\} = |G| - \nu(\overline{G}) + |H| = \mathrm{mr}_\nu(\overline{G}) + |H|,$$

so

$$\begin{aligned}\mathrm{mr}_\nu(G \vee H) + \mathrm{mr}_\nu(\overline{G \vee H}) &= \mathrm{mr}_\nu(G) + \mathrm{mr}_\nu(\overline{G}) + |H| \\ &\leq 2 + |G| + |H| = 2 + |G \vee H|,\end{aligned}$$

using the fact that G satisfies GCC_ν. Thus $G \vee H$ satisfies GCC_ν.

Now suppose that $\nu(\overline{H}) > \nu(\overline{G}) \geq 1$. Thus

$$|\overline{H}| - \mathrm{mr}_\nu(\overline{H}) > |\overline{G}| - \mathrm{mr}_\nu(\overline{G}).$$

Using reasoning similar to that above,

$$\begin{aligned}\mathrm{mr}_\nu(G \vee H) + \mathrm{mr}_\nu(\overline{G \vee H}) &= \mathrm{mr}_\nu(G) + \mathrm{mr}_\nu(\overline{H}) + |G| \\ &< \mathrm{mr}_\nu(G) + \mathrm{mr}_\nu(\overline{G}) + |H| \\ &\leq 2 + |G| + |H|,\end{aligned}$$

so again $G \vee H$ satisfies GCC_ν. □

A graph is said to be *decomposable* if it can be expressed as a sequence of joins and unions of isolated vertices (these graphs are also known as cographs). We also note that the complement of a decomposable graph is again decomposable.

COROLLARY 3.9. *If G is a decomposable graph, then G satisfies GCC_ν.*

3.2. GCC and $\mathbf{GCC_+}$ for joins of graphs. In this section for convenience we extend the definition of a graph to include a graph with no vertices, which will be denoted by $\emptyset$. By definition, $\mathrm{mr}(\emptyset) = \mathrm{mr}_+(\emptyset) = 0$.

If $G = \bigcup_{i=1}^{r} G_i$, where each G_i is connected, the subgraph $\breve{G} = \bigcup_{|G_i|>1} G_i$ is called the *core* of G, while $\ddot{G} = \bigcup_{|G_i|=1} G_i$ is called the *isolated part* of G. Note that if G is connected, then $G = \ddot{G}$ if and only if $|G| = 1$. Also if G has no isolated vertices then $G = \breve{G}$, and G is said to be *isolated free*, while if G only consists of one or more isolated vertices, then $\breve{G} = \emptyset$.

OBSERVATION 3.10. *Let G be a graph. Then* $\mathrm{mr}(G) = \mathrm{mr}(\breve{G})$, $\mathrm{mr}_+(G) = \mathrm{mr}_+(\breve{G})$.

The *join minimum rank* of $G \neq \emptyset$ is defined to be $\mathrm{jmr}(G) = \mathrm{mr}(K_1 \vee G)$ [4] and $\mathrm{jmr}(\emptyset) = 1$. Along similar lines, we define the notion of the join minimum rank within the setting of positive semidefinite matrices.

Definition 3.11. For any graph $G \neq \emptyset$, define

$$\mathrm{jmr}_+(G) = \mathrm{mr}_+(K_1 \vee G).$$

We also define $\mathrm{jmr}_+(\emptyset) = 1$.

The notion of join minimum rank is needed here, as the minimum rank of the join can be adversely affected if at least one of the graphs contains isolated vertices (see the next result for example). However, incorporating the join minimum rank then introduces a complication when it is applied to unions.

The following result is [4, Prop. 3.6] adapted to account for our definition of the minimum rank of $\emptyset$.

Proposition 3.12. *For any graph G,*

$$\mathrm{jmr}(G) = \begin{cases} \mathrm{mr}(G) & \textit{if and only if} \quad |\ddot{G}| = 0 \textit{ and } G \neq \emptyset, \\ \mathrm{mr}(G) + 1 & \textit{if and only if} \quad |\ddot{G}| = 1 \textit{ or } G = \emptyset, \\ \mathrm{mr}(G) + 2 & \textit{if and only if} \quad |\ddot{G}| \geqslant 2. \end{cases}$$

Lemma 3.13. *Let $G \neq \emptyset$ with $r \geq 0$ isolated vertices. Then*

1. $\mathrm{jmr}(G) = \mathrm{mr}(\breve{G}) + \min\{2, r\}$,
2. $\mathrm{jmr}_+(G) = \mathrm{mr}_+(\breve{G}) + r$.

Proof. The proof of (1) is a direct application of Proposition 3.12, as $\mathrm{mr}(G) = \mathrm{mr}(\breve{G})$. For (2), let $A \in \mathcal{S}((\breve{G} \cup \overline{K_r}) \vee K_1)$ be positive semidefinite and let B be the principal submatrix of A obtained by deleting the joined K_1. Then $\mathrm{rank}\, A \geq \mathrm{rank}\, B \geq \mathrm{mr}_+(\breve{G}) + r$, because B is a block diagonal matrix with positive diagonal entries associated with $\overline{K_r}$. By choosing a matrix of minimum semidefinite rank in $\mathcal{S}(\breve{G})$ and positive diagonal entries associated with $\overline{K_r}$, we construct a matrix $B' \in \mathcal{S}(\breve{G} \cup \overline{K_r})$ having rank $\mathrm{mr}_+(\breve{G}) + r$ that does not have a zero column, and it is straightforward to use B' to construct a matrix $A' \in \mathcal{S}((\breve{G} \cup \overline{K_r}) \vee K_1)$ of the same rank. ☐

We now move onto further notions of the core that will be relevant for the following discussion.

Definition 3.14. For any graph G, the *symmetric core*, denoted by $\widetilde{G}$ is defined as follows:

$$\widetilde{G} = \begin{cases} \breve{G} & \text{if } G \text{ has isolated vertices,} \\ \overline{(\breve{\overline{G}})} & \text{otherwise (i.e., complement of the core of the complement).} \end{cases}$$

Observe that $G = \widetilde{G}$ if and only if both G and $\overline{G}$ are isolated free. For a given graph G, we define inductively, the graphs: $G_0 = G$, and for $i = 1, 2, \ldots$, let $G_i = \widetilde{G_{i-1}}$.

Definition 3.15. The *inductive core* of G, denoted by $\breve{\breve{G}}$ is defined as

$$\breve{\breve{G}} = \bigcap_i G_i.$$

Evidently, $\breve{\breve{G}} = G_i$, where i is the first integer in which $G_i = \widetilde{G_i}$. Thus it follows that the core and the symmetric core of $\breve{\breve{G}}$ coincide with $\breve{\breve{G}}$ itself, that is, both $\breve{\breve{G}}$ and its complement are isolated free. Note, it may be the case that $\breve{\breve{G}}$ will have no vertices.

Definition 3.16. Let G be a graph. The *j-gap* and *j-gap*$_+$ of G are defined to be

$$\begin{aligned}
\operatorname{jgap}(G) &= \operatorname{jmr}(G) + \operatorname{jmr}(\overline{G}) - |G|, \\
\operatorname{jgap}_+(G) &= \operatorname{jmr}_+(G) + \operatorname{jmr}_+(\overline{G}) - |G|.
\end{aligned}$$

Clearly, we have that $\operatorname{jgap}(\overline{G}) = \operatorname{jgap}(G)$, for any graph G. Moreover, if $\operatorname{jgap}(G) \leq 2$, then since $\operatorname{jmr}(G) \geq \operatorname{mr}(G)$ it follows that G must satisfy GCC, and analogously for jgap_+ and GCC$_+$.

Lemma 3.17. *For any graph $G \neq \emptyset$:*

1. $\operatorname{jgap}(G) \leq \operatorname{jgap}(\breve{G})$,
2. $\operatorname{jgap}_+(G) \leq \operatorname{jgap}_+(\breve{G})$.

Proof. If G is isolated free then there is nothing to show, as $G = \breve{G}$. So, suppose $G = \breve{G} \cup \overline{K_r}$. Then $\overline{G} = \overline{\breve{G}} \vee K_r$, and we have $\operatorname{mr}(G) = \operatorname{mr}(\breve{G})$, $\operatorname{mr}(\breve{G}) \leq \operatorname{jmr}(\breve{G})$ (and equality holds unless $\breve{G} = \emptyset$). Finally, it follows that $\operatorname{jmr}(\overline{G}) = \operatorname{jmr}(\overline{\breve{G}})$, as $r \geq 1$ and $\operatorname{jmr}(X) = \operatorname{jmr}(X \vee K_1)$ for any graph X by Proposition 3.12.

Then we have

$$\operatorname{jmr}(G) = \operatorname{mr}(\breve{G}) + \min\{2, r\} \leq \operatorname{jmr}(\breve{G}) + r.$$

Hence

$$\begin{aligned}\operatorname{jgap}(G) &= \operatorname{jmr}(G) + \operatorname{jmr}(\overline{G}) - |G| \\ &\leq \operatorname{jmr}(\breve{G}) + r + \operatorname{jmr}(\overline{\breve{G}}) - (|\breve{G}| + r) \\ &= \operatorname{jgap}(\breve{G}).\end{aligned}$$

The proof of (2) is similar and is omitted here. ☐

Corollary 3.18. *For any graph $G \neq \emptyset$:*

1. $\operatorname{jgap}(G) \leq \operatorname{jgap}(\breve{\breve{G}})$,
2. $\operatorname{jgap}_+(G) \leq \operatorname{jgap}_+(\breve{\breve{G}})$.

Lemma 3.19. *If G is a graph such that $\breve{\breve{G}}$ satisfies GCC (respectively, GCC$_+$), then* $\operatorname{jgap}(G) \leq 2$ *(respectively,* $\operatorname{jgap}_+(G) \leq 2$*). Then G satisfies GCC (respectively, GCC$_+$).*

Proof. In view of Corollary 3.18 it is sufficient to show that $\operatorname{jgap}(\breve{\breve{G}}) \leq 2$. If $\breve{\breve{G}} = \emptyset$, then so is $\overline{\breve{\breve{G}}}$, and hence $\operatorname{jgap}(\emptyset) = 2$. So assume that $\breve{\breve{G}} \neq \emptyset$. Since $\breve{\breve{G}}$ and $\overline{\breve{\breve{G}}}$ are isolated free we have $\operatorname{jmr}(\breve{\breve{G}}) = \operatorname{mr}(\breve{\breve{G}})$ and $\operatorname{jmr}(\overline{\breve{\breve{G}}}) = \operatorname{mr}(\overline{\breve{\breve{G}}})$. Since $\breve{\breve{G}}$ satisfies GCC, we have $\operatorname{jgap}(\breve{\breve{G}}) \leq 2$. The proof for GCC$_+$ is similar. ☐

Lemma 3.20. *Let G and H be graphs. Then*

$$\operatorname{jmr}(G \cup H) \leq \operatorname{jmr}(G) + \operatorname{jmr}(H),$$

where the inequality can be strict. In the positive semidefinite case,

$$\operatorname{jmr}_+(G \cup H) \leq \operatorname{jmr}_+(G) + \operatorname{jmr}_+(H),$$

with equality provided both $G \neq \emptyset$ and $H \neq \emptyset$.

Proof. These results are immediate if either $G = \emptyset$ or $H = \emptyset$, so assume that both $G \neq \emptyset$ and $H \neq \emptyset$. In both cases we use Lemma 3.13. In the positive semidefinite case, assume $G \neq \emptyset$ has r_1 isolated vertices and $H \neq \emptyset$ has r_2 isolated vertices. Then $\operatorname{jmr}_+(G) = \operatorname{mr}_+(\breve{G}) + r_1$ and $\operatorname{jmr}_+(H) = \operatorname{mr}_+(\breve{H}) + r_2$, by Lemma 3.13. Another application of Lemma 3.13 yields

$$\begin{aligned}\operatorname{jmr}_+(G \cup H) &= \operatorname{mr}_+(\breve{G} \cup \breve{H}) + (r_1 + r_2) \\ &= \operatorname{mr}_+(\breve{G}) + \operatorname{mr}_+(\breve{H}) + r_1 + r_2 \\ &= \operatorname{jmr}_+(G) + \operatorname{jmr}_+(H).\end{aligned}$$

In the symmetric case, an inequality appears since

$$\operatorname{jmr}(G) = \operatorname{mr}(\breve{G}) + \min\{2, r\} \leq \operatorname{mr}(\breve{G}) + r,$$

whenever $r \geq 2$. To verify an instance of a strict inequality, consider $G = H = \overline{K_2}$. Then $\operatorname{jmr}(G) = \operatorname{jmr}(H) = 2$ and $\operatorname{jmr}(G \cup H) = 2$. □

LEMMA 3.21. *Suppose G is a given graph. Then* $\operatorname{jmr}(G) = 1$ *(*$\operatorname{jmr}_+(G) = 1$*) if and only if $G = K_r$ for some integer $r \geq 1$ or $G = \emptyset$.*

Proof. Since, in general, $\operatorname{jmr}(G) \geq \operatorname{mr}(G)$ (respectively, $\operatorname{jmr}_+(G) \geq \operatorname{mr}_+(G)$) it follows $\operatorname{jmr}(G) = 1$ if and only if $\operatorname{mr}(G) = 0$ or 1 (respectively, $\operatorname{jmr}_+(G) = 1$ if and only if $\operatorname{mr}_+(G) = 0$ or 1). The conclusion, then, readily follows. □

Before we come to our main results on the join of graphs, we recall the following fact that can be found in [4] and deduced from the work in [6]. If the minimum rank of a graph G is at most 2, then G must be a decomposable graph. For more information, on the minimum rank of the joins of graphs and of decomposable graphs, see [4]. In particular, let $G = \bigvee_{i=1}^{r} G_i$ be a decomposable graph. Then G is said to be *anomalous* if

1. for each i, $\operatorname{jmr}(G_i) \leqslant 2$; and
2. $K_{3,3,3}$ is a subgraph of G.

In particular, in a non-anomalous decomposable graph G with $\operatorname{mr}(G) \leq 2$, there are at most two i for which $|G_i| \geqslant 3$ and $G_i = \ddot{G}_i$.

We now need to state a result that was originally used in [4, Lemma 3.7] for the case of inertially balanced graphs, but in fact this result holds under more relaxed conditions.

LEMMA 3.22. *Let $G \neq \emptyset$ be a graph. There exists $A \in \mathcal{S}(G)$ such that A has a nonzero (h, k)-representation with $h + k = \operatorname{jmr}(G)$. There exists $A \in \mathcal{S}_+(G)$ such that* $\operatorname{rank} A = \operatorname{jmr}_+(G)$ *and A has a nonzero* $(\operatorname{jmr}_+(G), 0)$*-representation.*

Proof. We only provide a proof in the positive semidefinite case, as the argument for the indefinite case is identical to the one provided in [4, Lemma 3.7].

For the positive semidefinite case, suppose $\operatorname{jmr}_+(G) = \operatorname{mr}_+(G)$. Then G has no isolated vertices and any matrix $A \in \mathcal{S}_+(G)$ with $\operatorname{rank} A = \operatorname{mr}_+(G)$ has a $(\operatorname{jmr}_+(G), 0)$-representation. On the other hand, if $\operatorname{jmr}_+(G) = \operatorname{mr}_+(\breve{G}) + r$ where $r \geq 1$, then $G = \breve{G} \cup \overline{K_r}$. Let $\begin{bmatrix} P \end{bmatrix}$ be a nonzero $(\operatorname{mr}_+(\breve{G}), 0)$-representation for any optimal matrix in $\mathcal{S}(\breve{G})$. Then

$$\begin{bmatrix} P & 0 \\ 0 & I_r \end{bmatrix}$$

is a nonzero $(\mathrm{jmr}_+(G), 0)$-representation for a matrix in $\mathcal{S}_+(G)$. □

A related result on the join of two graphs appears in [14] in the context of Hermitian positive semidefinite matrices. Since the analysis in Lemmas 3.2 and 3.22 requires only working over the reals, we have the next result as a consequence.

Corollary 3.23. *Let $G \neq \emptyset$ and $H \neq \emptyset$ be two disjoint graphs. Then*

$$\mathrm{jmr}_+(G \vee H) = \mathrm{mr}_+(G \vee H) = \max\{\mathrm{jmr}_+(G), \mathrm{jmr}_+(H)\}.$$

Proof. Let $m = \max\{\mathrm{jmr}_+(G), \mathrm{jmr}_+(H)\}$. Then by Lemma 3.22 there exist $A \in \mathcal{S}(G), B \in \mathcal{S}(H)$ having nonzero $(m, 0)$-representations, so by Lemma 3.2 there is a $(m, 0)$-representation a matrix in $\mathcal{S}_+(G \vee H)$. Thus $\mathrm{mr}_+(G \vee H) \leq m$, but since G and H are induced subgraphs of $G \vee H$, $\mathrm{mr}_+(G \vee H) \geq m$ also. The result for join minimum rank follows from the fact that $G \vee H$ does not have isolated vertices. □

Theorem 3.24. *Let G and H be two disjoint graphs with $G \neq \emptyset$ and $H \neq \emptyset$. Then:*

1. *If* $\mathrm{jmr}(G) \geq \mathrm{jmr}(H) \geq 1$ *and*
   (a) *if* $\mathrm{jmr}(H) \geq 3$, *then* $\mathrm{jmr}(G \vee H) \leq \mathrm{jmr}(G) + \mathrm{jmr}(H) - 2$;
   (b) *if* $\mathrm{jmr}(H) \leq 2$, *then* $\mathrm{jmr}(G \vee H) \leq \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1$.
2. *If* $\mathrm{jmr}_+(G) \geq \mathrm{jmr}_+(H) \geq 1$ *and*
   (a) *if* $\mathrm{jmr}_+(H) \geq 2$, *then* $\mathrm{jmr}_+(G \vee H) \leq \mathrm{jmr}_+(G) + \mathrm{jmr}_+(H) - 2$;
   (b) *if* $\mathrm{jmr}_+(H) = 1$, *then* $\mathrm{jmr}_+(G \vee H) \leq \mathrm{jmr}_+(G) + \mathrm{jmr}_+(H) - 1$.

Proof. For 1(a), suppose $\mathrm{jmr}(H) \geq 3$. By Lemma 3.22 we can choose $A \in \mathcal{S}(G)$ such that A has a nonzero (h_G, k_G)-representation, with $h_G + k_G = \mathrm{jmr}(G)$. Since $\mathrm{jmr}(G) \geq 3$, by replacing A by $-A$ if necessary, we may assume $h_G \geq 2$, so $k_G \leq \mathrm{jmr}(G) - 2$. Similarly, choose $B \in \mathcal{S}(H)$ having a nonzero (h_H, k_H)-representation, with $h_H + k_H = \mathrm{jmr}(H)$, $h_H \geq 2$, and $k_H \leq \mathrm{jmr}(H) - 2$.

Define $h = \max\{h_G, h_H\}$ and $k = \max\{k_G, k_H\}$. Then, by padding with zero rows as needed, there exist nonzero (h, k)-representations for A and B, respectively. Then, by Lemma 3.2, we may construct a symmetric matrix in $\mathcal{S}(G \vee H)$ with rank at most $h + k$. Thus it follows that

$$\mathrm{jmr}(G \vee H) = \mathrm{mr}(G \vee H) \leq h + k.$$

Observe that among the four possible sums of $h + k$, the maximum is always bounded above by $\mathrm{jmr}(G) + \mathrm{jmr}(H) - 2$, as desired.

For 1(b), consider first the case when $\mathrm{jmr}(G) \geq 3$ and $\mathrm{jmr}(H) = 2$. As with the argument applied in the case above, choose $A \in \mathcal{S}(G)$ with a nonzero (h_G, k_G)-

representation in which $h_G + k_G = \mathrm{jmr}(G)$, and $h_G \geq 2$, $k_G \leq \mathrm{jmr}(G) - 2$; and choose $B \in \mathcal{S}(H)$ having a nonzero (h_H, k_H)-representation, with $h_H \geq 1$, $k_H \leq 1$, and $h_H + k_H = 2$. As above, we can construct, by Lemma 3.2 a matrix in $\mathcal{S}(G \vee H)$ with rank at most $h + k$, where $h = \max\{h_G, h_H\} = h_G$ and $k = \max\{k_G, k_H\}$. It follows that

$$\mathrm{jmr}(G \vee H) = \mathrm{mr}(G \vee H) \leq h + k \leq \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1.$$

Under 1(b), the next case to consider is $\mathrm{jmr}(G) \geq 3$ and $\mathrm{jmr}(H) = 1$ Then, as in the previous case, we may choose $A \in \mathcal{S}(G)$ having a nonzero (h_G, k_G)-representation, with $h_G + k_G = \mathrm{jmr}(G)$, and with $h_G \geq 2$, $k_G \leq \mathrm{jmr}(G) - 2$. Further, since $\mathrm{jmr}(H) = 1$, $H = K_r$ for some $r \geq 1$ (Lemma 3.21), so let $B \in \mathcal{S}(H)$ with $i_+(B) = 1$ and $i_-(B) = 0$. Applying Lemma 3.2, we can construct a matrix in $\mathcal{S}(G \vee H)$ with rank at most $h_G + h_K$. Then we have

$$\mathrm{mr}(G \vee H) = \mathrm{jmr}(G \vee H) = \mathrm{jmr}(G) = \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1.$$

The next case to consider under 1(b) is $\mathrm{jmr}(G) = 2$ and $\mathrm{jmr}(H) = 2$. Then both G and H are decomposable and hence so is $G \vee H$. By Theorem 4.5 [4] we have

$$\begin{aligned} \mathrm{jmr}(G \vee H) &\leq \max\{\mathrm{jmr}(G), \mathrm{jmr}(H)\} + 1 \\ &= \mathrm{jmr}(G) + 1 \\ &= \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1. \end{aligned}$$

The final case under 1(b) is $\mathrm{jmr}(G) \leq 2$ and $\mathrm{jmr}(H) = 1$. Then, again, both G and H are decomposable as is $G \vee H$. The graph $K_{3,3,3}$ is not induced in G because $\mathrm{mr}(G) \leq \mathrm{jmr}(G) \leq 2 < 3 = \mathrm{mr}(K_{3,3,3})$, so it is not induced in $G \vee H$, as H is a complete graph. Thus $G \vee H$ is not anomalous. Hence in this case we have

$$\begin{aligned} \mathrm{jmr}(G \vee H) &\leq \max\{\mathrm{jmr}(G), \mathrm{jmr}(H)\} \\ &= \mathrm{jmr}(G) \\ &= \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1, \end{aligned}$$

where, again, the first inequality follows from [4, Thm. 4.5].

For (2), the positive semidefinite case, the arguments are very similar, and since we require $i_+ \geq 2$ to apply Lemma 3.2 the inequality conditions in 2(a) and 2(b) have been reduced by 1 as compared with item 1. ∎

Specializing to the case of decomposable graphs, we have the next result, which not only demonstrates that they satisfy GCC (or GCC_+), but they satisfy a slightly stronger condition.

THEOREM 3.25. *If $G \neq \emptyset$ is a decomposable graph, then* $\mathrm{jgap}(G) \leq 1$ *(respectively,* $\mathrm{jgap}_+(G) \leq 1$*).*

Proof. The proof is by induction on the order of the decomposable graph. Observe that if $G = K_1$, then $\mathrm{jmr}(G) = \mathrm{jmr}(\overline{G}) = |G| = 1$. Hence $\mathrm{jgap}(G) = 1$ for the base case. Now, consider two arbitrary decomposable graphs G and H, each with jgap at most one. For the decomposable graph $G \vee H$, assume first that $G \vee H$ is not anomalous and $\mathrm{jmr}(G) \geq \mathrm{jmr}(H)$. In this case,

$$\begin{aligned}
\mathrm{jgap}(G \vee H) &= \mathrm{jmr}(G \vee H) + \mathrm{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|)\} \\
&\leq \max\{\mathrm{jmr}(G), \mathrm{jmr}(H)\} + \mathrm{jmr}(\overline{G}) + \mathrm{jmr}(\overline{H}) - (|G| + |H|) \\
&= \mathrm{jmr}(G) + \mathrm{jmr}(\overline{G}) - |G| + (\mathrm{jmr}(\overline{H}) - |H|) \\
&\leq \mathrm{jmr}(G) + \mathrm{jmr}(\overline{G}) - |G| \\
&= \mathrm{jgap}(G) \leq 1,
\end{aligned}$$

where the first inequality follows from [4, Thm. 4.5] and Lemma 3.20. If, on the other hand, $G \vee H$ is anomalous, then $\mathrm{jmr}(G \vee H) = 3$ by [4, Thm. 4.5], and

$$\begin{aligned}
\mathrm{jgap}(G \vee H) &= \mathrm{jmr}(G \vee H) + \mathrm{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|) \\
&\leq 3 + \mathrm{jmr}(\overline{G}) - |G| + \mathrm{jmr}(\overline{H}) - |H|.
\end{aligned}$$

Observe that for any graph X, $\mathrm{jmr}(X) = |X|$ if and only if $X = K_1$ or $X = K_1 \cup K_1$. If both $\mathrm{jmr}(\overline{G}) < |\overline{G}| = |G|$ and $\mathrm{jmr}(\overline{H}) < |\overline{H}| = |H|$, then $\mathrm{jgap}(G \vee H) \leq 1$. Further, since $G \vee H$ is anomalous, it is not possible for both equalities $\mathrm{jmr}(\overline{G}) = |\overline{G}|$ and $\mathrm{jmr}(\overline{H}) = |\overline{H}|$ to hold. Thus, without loss of generality, assume $\mathrm{jmr}(\overline{H}) = |\overline{H}|$ and hence G must itself be anomalous (and decomposable). In this case $\mathrm{jmr}(G) = 3$, and hence $\mathrm{jgap}(G \vee H) = \mathrm{jgap}(G) \leq 1$, by induction.

Since the parameter jgap is symmetric with respect to complementation, the case of the union of two decomposable graphs follows trivially. This completes the proof, as any decomposable graph can be decomposed as a union or join of two decomposable graphs.

The argument in the positive semidefinite case can be proved in a similar manner in the nonanomalous case by using Corollary 3.23. ∎

We are now in a position to state and prove the main results of this subsection on the join and union of graphs and the GCC.

Theorem 3.26. *Suppose G and H are two graphs. Then*

1. *if* $\mathrm{jgap}(G)$ *and* $\mathrm{jgap}(H)$ *are both at most two, then*
$$\mathrm{jgap}(G \vee H) \leq 2;$$
2. *if* $\mathrm{jgap}_+(G)$ *and* $\mathrm{jgap}_+(H)$ *are both at most two, then*
$$\mathrm{jgap}_+(G \vee H) \leq 2.$$

Proof. First, we assume without loss of generality, that $\mathrm{jmr}(G) \geq \mathrm{jmr}(H)$. For (1), suppose $\mathrm{jgap}(G), \mathrm{jgap}(H) \leq 2$. We separate the argument into two cases: $\mathrm{jmr}(H) \geq 3$ and $\mathrm{jmr}(H) \leq 2$.

If $\mathrm{jmr}(H) \geq 3$, then we have

$$\begin{aligned}
\mathrm{jgap}(G \vee H) &= \mathrm{jmr}(G \vee H) + \mathrm{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|)\} \\
&\leq \mathrm{jmr}(G) + \mathrm{jmr}(H) - 2 + \mathrm{jmr}(\overline{G}) + \mathrm{jmr}(\overline{H}) - (|G| + |H|) \\
&= \mathrm{jgap}(G) + \mathrm{jgap}(H) - 2 \\
&\leq 2 + 2 - 2 = 2,
\end{aligned}$$

where the first inequality follows from 1(a) of Theorem 3.24 and Lemma 3.20.

If, otherwise, $\mathrm{jmr}(H) \leq 2$, then H is decomposable, and so by Theorem 3.25 we have $\mathrm{jgap}(H) \leq 1$. Thus it follows that,

$$\begin{aligned}
\mathrm{jgap}(G \vee H) &= \mathrm{jmr}(G \vee H) + \mathrm{jmr}(\overline{G} \cup \overline{H}) - (|G| + |H|)\} \\
&\leq \mathrm{jmr}(G) + \mathrm{jmr}(H) - 1 + \mathrm{jmr}(\overline{G}) + \mathrm{jmr}(\overline{H}) - (|G| + |H|) \\
&= \mathrm{jgap}(G) + \mathrm{jgap}(H) - 1 \\
&\leq 2 + 1 - 1 = 2.
\end{aligned}$$

The first inequality above follows from 1(b) of Theorem 3.25 and Lemma 3.20.

The positive semidefinite case follows in a similar manner. □

The following are immediate consequences of the above results.

Corollary 3.27. *If G and H are two given graphs, and both of their inductive cores satisfy GCC (respectively, GCC_+), then $G \vee H$ and $G \cup H$ satisfy GCC (respectively, GCC_+).*

Proof. If G and H are two given graphs such that both of their inductive cores satisfy GCC (or GCC_+), then $\mathrm{jgap}(\breve{\breve{G}}) \leq 2$ and $\mathrm{jgap}(\breve{\breve{H}}) \leq 2$ (if $\breve{\breve{G}} \neq \emptyset$, then $\mathrm{jmr}(\breve{\breve{G}}) =$

$\mathrm{mr}(\breve{\breve{G}})$, and if $\breve{\breve{G}} = \emptyset$, then $\mathrm{jgap}(\breve{\breve{G}}) = 2$). By Lemma 3.18 both $\mathrm{jgap}(G) \leq 2$ and $\mathrm{jgap}(H) \leq 2$ and similarly for the positive semidefinite case. Thus an application of Theorem 3.26 implies that $G \vee H$ satisfies GCC (GCC_+). The fact that the union of G and H satisfy GCC (or GCC_+) follows from complementation and the fact that both the hypothesis and conclusion are symmetric under the operation of taking complements. □

Corollary 3.28. *If $\delta(G) \geq |G| - 3$, then G satisfies GCC_+ (and hence GCC), where $\delta(G)$ represents the minimum degree of G.*

Proof. Observe that if $\delta(G) \geq |G| - 3$, then $\overline{G}$ is a disjoint union of cycles and paths. Then apply Corollary 3.27, as the inductive cores of both paths and cycles can easily be seen to satisfy GCC (or GCC_+). □

Requiring the inductive core to satisfy GCC seems critical. Suppose G is a graph that does not satisfy GCC. Without loss of generality, we may assume that $G = \breve{\breve{G}}$ (by Lemma 3.19). For the purposes of this argument, we actually need to assume that

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) = |G| + 4$$

(if it is larger than 4, the argument below can be modified). Define the new graph $G' = G \cup \overline{K_2}$. Since G has no isolated part ($G = \breve{\breve{G}}$), by Proposition 3.12 we have $\mathrm{mr}(G') = \mathrm{mr}(G)$ and $\mathrm{mr}(\overline{G'}) = \mathrm{mr}(\overline{G})$. Then, it follows that G' satisfies GCC. So G' is a graph that satisfies GCC but its inductive core does not. Now let $H = \{w\}$, and form $\Gamma = G' \vee H$. Again, applying Proposition 3.12 shows that $\mathrm{mr}(\Gamma) = \mathrm{mr}(G') + 2$ and $\mathrm{mr}(\overline{\Gamma}) = \mathrm{mr}(\overline{G'})$. Thus we may conclude that Γ does not satisfy GCC.

4. Conclusion. We close this work by formulating some basic necessary conditions on a potential counterexample for each conjecture. We begin with a discussion of graphs that attain low minimum rank. Since for every graph H, $\mathrm{mr}(H)$, $\mathrm{mr}_+(H), \mathrm{mr}_\nu(H) \leq |H| - 1$, GCC (respectively, GCC_+, GCC_ν) is valid for graphs G that satisfy $\mathrm{mr}(G) \leq 3$ ($\mathrm{mr}_+(G) \leq 3, \mathrm{mr}_\nu(G) \leq 3$). The low minimum rank case argument can be pushed a little further (as was done in [15] for GCC).

Proposition 4.1. *Suppose* $\mathrm{mr}(G) \leq 4$ *(respectively,* $\mathrm{mr}_+(G) = 4$, $\mathrm{mr}_\nu(G) = 4$*). Then G satisfies* GCC *(*GCC_+, GCC_ν*).*

Proof. Assume that $\mathrm{mr}(G) \leq 4$ and G does not satisfy GCC (GCC_+, GCC_ν). Then it follows that $\mathrm{mr}(\overline{G}) = n - 1$ ($\mathrm{mr}_+(\overline{G}) = \mathrm{mr}_\nu(\overline{G}) = n - 1$). Hence $\overline{G} = P_n$ ($\overline{G}$ is a tree, or a forest) (see [13, 16]). However, paths (trees or forests) on n vertices satisfy GCC (GCC_+, GCC_ν), which is a contradiction. □

Hence it follows that if GCC, GCC_+, or GCC_ν fails for a given graph G, then G

must satisfy:

$$5 \leq \mathrm{mr}(G) \leq \mathrm{mr}_+(G) \leq \mathrm{mr}_\nu(G),$$

and

$$5 \leq \mathrm{mr}(\overline{G}) \leq \mathrm{mr}_+(\overline{G}) \leq \mathrm{mr}_\nu(\overline{G}).$$

Concerning the GCC, by Corollaries 2.10 and 2.11 and Proposition 4.1 it follows that the first possible counterexample for GCC is a graph on 11 vertices that satisfies $\mathrm{mr}(G) = \mathrm{mr}(\overline{G}) = 7$.

Similarly for GCC_+ and GCC_ν we may deduce from Corollaries 2.7 and 2.8 and Proposition 4.1 that a first potential counterexample for GCC_ν or GCC_+ would be a graph G on 9 vertices that satisfies $\nu(G) = \nu(\overline{G}) = 3$ or $\mathrm{M}_+(G) = \mathrm{M}_+(\overline{G}) = 3$.

Furthermore, from the work in Section 3.1 we may conclude that a minimal counterexample for GCC_ν must be a graph for which both it and its complement are connected. A similar statement can be made for the conjectures GCC and GCC_+ involving inductive cores, but the actual details are omitted here.

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REFERENCES

[1] N. Achuthan, N.R. Achuthan, and L. Caccetta, On the Nordhaus-Gaddum class problems, Combinatorial Mathematics and combinatorial computing, Vol. 2 (Brisbane 1989). *Australas. J. Combin. 2*, 5–27, 1990.

[2] American Institute of Mathematics workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns," held October 23-27, 2006 in Palo Alto, CA. Workshop webpage: `http://aimath.org/pastworkshops/matrixspectrum.html`.

[3] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.

[4] F. Barioli and S.M. Fallat. On the minimum rank of the join of graphs and decomposable graphs, *Linear Algebra and its Applications*, 421: 252–263, 2007.

[5] F. Barioli, S.M. Fallat, and L. Hogben. A variant on the graph parameters of Colin de Verdière: Implications to the minimum rank of graphs. *Electronic Journal of Linear Algebra*, 13: 387–404, 2005.

[6] W. Barrett, H. van der Holst and R. Loewy. Graphs whose minimal rank is two. *Electronic Journal of Linear Algebra*, 11: 258–280, 2004.

[7] R. B. Borie, R. G. Parker, and C. A. Tovey. Recursively constructed graphs, in *Handbook of Graph Theory*, J. L. Gross and J. Yellen, Eds. CRC Press, Boca Raton, 2004, pp. 99–118.

[8] R. Brualdi, L. Hogben and B. Shader "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns Final Report: Mathematical Results (revised)" Workshop webpage: `http://aimath.org/pastworkshops/matrixspectrum.html`.

[9] Y. Colin de Verdière. On a new graph invariant and a criterion for planarity. In *Graph Structure Theory*. Contemporary Mathematics 147, American Mathematical Society, Providence,137–147, 1993.

[10] Y. Colin de Verdière, Multiplicities of eigenvalues and tree-width of graphs, *J. Combin. Theory Ser. B* 74: 121–146, 1998.

[11] L. DeLoss, J. Grout, L. Hogben, T. McKay, J. Smith, and G. Tims, Techniques for determining the minimum rank of a small graph. *Linear Algebra and its Applications*, 432: 2995–3001, 2010.

[12] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear Algebra and its Applications*, 426: 558–582, 2007.

[13] M. Fiedler. A characterization of tridiagonal matrices. *Linear Algebra and its Applications*, 2: 191–197, 1969.

[14] P. Hackney, B. Harris, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, Linearly independent vertices and the minimum semidefinite rank. *Linear Algebra and its Applications*, 431: 1105–1115, 2009.

[15] L. Hogben. Orthogonal representations, minimum rank, and graph complements. *Linear Algebra and its Applications*, 428: 2560-2568, 2008.

[16] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and its Applications*, 375: 1-11, 2003.

[17] H. van der Holst. Three-connected graphs whose maximum nullity is at most three. *Linear Algebra and its Applications* 429: 625–632, 2007.

[18] H. van der Holst, L. Lovász, and A. Schrijver, The Colin de Verdière graph parameter, in: *Graph Theory and Computational Biology (Balatonlelle, 1996)*, Bolyai Soc. Math. Stud., 7, János Bolyai Math. Soc., Budapest, 1999, pp. 29–85.

[19] H. van der Holst and J. Sinkovic, The minimum semidefinite rank of the complement of k-trees. Accepted to *Linear Algebra and its Applications*.

[20] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree *Combinatorica* 4: 307–316, 1984.

[21] A. Kotlov, L. Lovász, S. Vempala. The Colin de Verdère number and sphere representations of a graph. *Combinatorica* 17: 483–521, 1997.

[22] L. Lovász, M. Saks and A. Schrijver. Orthogonal representations and connectivity of graphs. *Linear Algebra and its Applications* 114/115: 439–454, 1989.

[23] L. Lovász, M. Saks and A. Schrijver. A correction: "Orthogonal representations and connectivity of graphs." *Linear Algebra and its Applications* 313: 101–105, 2000.