Minimum rank and maximum nullity of tree sign patterns

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Dedicated to Pete Stewart on his 70th birthday.

Abstract

The minimum rank of a sign pattern matrix is defined to be the smallest possible rank over all real matrices having the given sign pattern. Maximum nullity of a sign pattern is the largest possible nullity over the same set of matrices, and is equal to the number of columns minus the minimum rank of the sign pattern. Definitions of various graph parameters used in [1] to bound maximum nullity of a zero-nonzero pattern, including path cover number and edit distance, are extended to sign patterns, and the SNS number is introduced to usefully generalize triangle number to sign patterns. It is shown that for tree sign patterns (that need not be combinatorially symmetric), minimum rank is equal to SNS number, and maximum nullity, path cover number and edit distance are equal, providing a method to compute minimum rank for tree sign patterns that are not combinatorially symmetric.

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1 Introduction

The *minimum rank problem for a sign pattern* asks us to determine the minimum rank among all real matrices whose pattern of signs is described by a given sign pattern matrix (that need not be square). This is a variant on the *symmetric minimum rank problem for a simple graph*, which asks us to to determine the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G (the diagonal of the matrix is free), and the *asymmetric minimum rank problem*, which asks us to to determine the minimum rank among all real matrices whose zero-nonzero pattern of entries is described by a given digraph or (possibly rectangular) zero-nonzero pattern. The symmetric minimum rank problem arose from the study of possible eigenvalues of real symmetric matrices described by a graph and has received considerable attention over the last ten years (see [5] and references

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therein). Recently minimum rank problems for digraphs or zero-nonzero patterns have been receiving attention (see, for example, [1, 3, 11]).

Minimum rank problems have found application to the study of communication complexity in computer science. Most of these connections involve the minimum rank of a sign pattern rather than a graph or digraph. For example, Forster [6] establishes a lower bound on the minimum rank of a sign pattern having no zero entries and uses this to establish a linear lower bound on unbounded error probabilistic communication complexity. Thus it is desirable to extend results for the minimum rank of graphs and digraphs to sign patterns.

Work on minimum rank problems began with trees. The minimum rank problem was solved for simple trees in [12] and [10], for combinatorially symmetric tree sign patterns and trees that allow loops in [4], for ditrees in [1], and for simple ditrees (matrices with unconstrained diagonal) in [8]. In this paper we adapt the methods of [1] to extend the solution of the minimum rank problem for combinatorially symmetric tree sign patterns given in [4] to all tree sign patterns. In some cases new techniques are needed; in others the proof can be adapted from the proof of the corresponding result in [1]. When the proof can be adapted, the corresponding result is cited and the proof is omitted, but may be found in the longer version of this paper [9].

A *sign pattern matrix* (or *sign pattern* for short) is a (rectangular) matrix having entries in $\{+,-,0\}$. For a real matrix A, $\mathrm{sgn}(A)$ is the sign pattern having entries that are the signs of the corresponding entries in A. If $\mathcal{Y}$ is an $n \times n$ sign pattern, the *sign pattern class* (or *qualitative class*) of $\mathcal{Y}$, denoted $\mathcal{Q}(\mathcal{Y})$, is the set of all $A \in \mathbb{R}^{n\times n}$ such that $\mathrm{sgn}(A) = \mathcal{Y}$.

The *minimum rank* of an $m \times n$ sign pattern $\mathcal{Y}$ is

$$\mathrm{mr}(\mathcal{Y}) = \min\{\mathrm{rank}(A) : A \in \mathcal{Q}(\mathcal{Y})\},$$

and the *maximum nullity* of $\mathcal{Y}$ is

$$\mathrm{M}(\mathcal{Y}) = \max\{\mathrm{null}(A) : A \in \mathcal{Q}(\mathcal{Y})\}.$$

Clearly $\mathrm{mr}(\mathcal{Y}) + \mathrm{M}(\mathcal{Y}) = n$.

It is traditional in the study of sign patterns to say that a sign pattern $\mathcal{Y}$ *requires* property P if every matrix in $\mathcal{Q}(\mathcal{Y})$ has property P and to say that $\mathcal{Y}$ *allows* property P if there exists a matrix in $\mathcal{Q}(\mathcal{Y})$ that has property P. In our study of minimum rank, we are interested in sign patterns that require nonsingularity, or equivalently, that do not allow singularity. A square sign pattern $\mathcal{Y}$ is *sign nonsingular* (SNS) if $\mathcal{Y}$ requires nonsingularity, i.e., if every matrix $A \in \mathcal{Q}(\mathcal{Y})$ is nonsingular. A sign pattern $\mathcal{Y}$ has *signed determinant* if the sign of the determinant of A is the same for every matrix $A \in \mathcal{Q}(\mathcal{Y})$. If the sign pattern $\mathcal{Y}$ has signed determinant $+$ or signed determinant $-$, then clearly $\mathcal{Y}$ is an SNS sign pattern, and the converse is also true [2].

Let $\mathcal{Y} = [\psi_{ij}]$ be an $n \times n$ sign pattern. The *generic matrix* $Y_{\mathcal{Y}}$ of $\mathcal{Y}$ is the matrix having i, j entry $\psi_{ij} y_{ij}$, where y_{ij} are independent indeterminates, $i, j = 1, \ldots, n$. It is well-known that $\mathcal{Y}$ is an SNS sign pattern if and only if at least one of the $n!$ terms in the standard expansion of the determinant of $Y_{\mathcal{Y}}$ is nonzero and all nonzero terms have the same sign [2]. Thus one can determine whether $\mathcal{Y}$ is an SNS sign pattern by evaluating $\det Y_{\mathcal{Y}}$.

A *graph* is simple (no loops or multiple edges) and is denoted $G = (V_G, E_G)$ where V_G and E_G are the sets of vertices and edges (two element subsets of vertices) of G, whereas a *digraph* allows loops (but not multiple copies of the same arc) and is denoted by $\Gamma = (V_\Gamma, E_\Gamma)$ where

V_Γ and E_Γ are the sets of vertices and arcs (ordered pairs of vertices) of Γ; in both cases, the set of vertices is finite and nonempty. If $\mathcal{Y} = [\psi_{ij}]$ is an $n \times n$ sign pattern, the *signed digraph of* $\mathcal{Y}$, $\Gamma^\pm(\mathcal{Y})$, is obtained from the digraph $(\{1,\dots,n\}, \{(i,j) : \psi_{ij} \neq 0\})$ by attaching the sign $\psi_{ij} \in \{+,-\}$ to the arc (i,j). We use the term *(signed) digraph* to mean a digraph or signed digraph.

A *path* is a graph or (signed) digraph $P_k = (\{v_1,\dots,v_k\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1,\dots,k-1\}$ or $E = \{(v_i, v_{i+1}) : i = 1,\dots,k-1\}$. A *cycle* is a graph or (signed) digraph $C_k = (\{v_1,\dots,v_k\}, E)$ such that $E = \{\{v_i, v_{i+1}\} : i = 1,\dots,k-1\} \cup \{\{v_k, v_1\}\}$ or $E = \{(v_i, v_{i+1}) : i = 1,\dots,k-1\} \cup \{(v_k, v_1)\}$; the order in which the vertices appear in the cycle is denoted $(v_1,\dots,v_k)$. A *generalized cycle* is the disjoint union of one or more cycles. The *length* of a path or cycle is the number of edges or arcs. Note that $(\{v\}, \{(v,v)\})$ is a digraph cycle of length one and $(\{v,w\}, \{(v,w),(w,v)\})$ is a digraph cycle of length two, whereas the minimum length of a graph cycle is three. Let $\mathcal{Y}$ be a sign pattern, let $B = [b_{ij}]$ be a matrix having sign pattern $\mathcal{Y}$ (possibly the generic matrix of $\mathcal{Y}$). The *cycle product* in B of a cycle $(v_1,\dots,v_k)$ in $\Gamma^\pm(\mathcal{Y})$ is $b_{v_1,v_2} \dots b_{v_{k-1},v_k} b_{v_k,v_1}$, and a *generalized cycle product* in B is the product of the cycle products corresponding to the cycles in the generalized cycle.

Let Γ be a (signed) digraph. To *reverse* arc (v,w) means to replace it by arc (w,v) (assigning the sign of (v,w) to (w,v) in the case of a signed digraph). The (signed) digraph obtained from Γ by reversing all the arcs of Γ will be denoted by Γ^T. Since for any $A \in \mathbb{R}^{n\times n}$, $\Gamma^\pm(A^T) = \Gamma^\pm(A)^T$ and $\operatorname{rank}(A^T) = \operatorname{rank}(A)$, $\operatorname{mr}(\Gamma^T) = \operatorname{mr}(\Gamma)$. A (signed) digraph Γ is *symmetric* if $\Gamma = \Gamma^T$ (note this is equality, not isomorphism). A (signed) digraph Γ is *combinatorially symmetric* if the digraph obtained by ignoring the signs is symmetric, and a sign pattern $\mathcal{Y}$ is *combinatorially symmetric* if $\Gamma^\pm(\mathcal{Y})$ is combinatorially symmetric.

A vertex w is an *out-neighbor* of vertex u in Γ if (u,w) is an arc of Γ. Note that v is an out-neighbor of itself if and only if the loop (v,v) is an arc of Γ. In a combinatorially symmetric (signed) digraph, an out-neighbor is called a *neighbor*.

For a (signed) digraph Γ, the *underlying graph* of Γ is the graph G obtained from Γ by deleting loops and then replacing every arc (v,w) or pair of arcs $(v,w),(w,v)$ by the edge $\{v,w\}$ (and by ignoring the signs if Γ is a signed digraph). A *tree* is a connected acyclic graph and a *(signed) directed tree* or *(signed) ditree* is a (signed) digraph whose underlying graph is a tree. A square sign pattern $\mathcal{Y}$ is a *tree sign pattern* if $\Gamma^\pm(\mathcal{Y})$ is a signed ditree.

For $\mathcal{Y}$ an $m \times n$ sign pattern and $R \subseteq \{1,\dots,m\}, C \subseteq \{1,\dots,n\}$, define the *subpattern* $\mathcal{Y}[R|C]$ to be the submatrix of $\mathcal{Y}$ lying in the rows that have indices in R and columns that have indices in C. This usage of the term "subpattern" is consistent with the term "submatrix," but differs from the way the term "subpattern" is often used in sign pattern literature, where it means a pattern of the same size as the original but with some nonzero entries replaced by 0s. In a square sign pattern, a *principal subpattern* is a subpattern of the form $\mathcal{Y}[R|R]$; such subpattern is denoted $\mathcal{Y}[R]$. For a (signed) digraph $\Gamma = (V_\Gamma, E_\Gamma)$ and $R \subseteq V_\Gamma$, the *induced subdigraph* $\Gamma[R]$ is the digraph with vertex set R and arc set $\{(v,w) \in E_\Gamma \mid v,w \in R\}$. The induced subdigraph $\Gamma^\pm(\mathcal{Y})[R]$ is naturally associated with the signed digraph of the the principal subpattern $\Gamma^\pm(\mathcal{Y}[R])$. The (signed) subdigraph induced by $\overline{R} = \{1,\dots,n\} \setminus R$ is usually denoted by $\Gamma - R$, or in the case R is a single vertex v, by $\Gamma - v$. In conformity with graph notation, we denote the subpattern $\mathcal{Y}[\overline{R}]$ by $\mathcal{Y} - R$.

There is a one-to-one correspondence between square sign patterns and signed digraphs. The *associated* sign pattern of a signed digraph Γ is the sign pattern $\mathcal{Y}$ such that $\Gamma^{\pm}(\mathcal{Y}) = \Gamma$. We apply digraph terminology to square sign patterns and vice versa. Specifically, a *component* of a square sign pattern $\mathcal{Y}$ is a principal subpattern $\mathcal{Y}[R]$ of $\mathcal{Y}$ such that $\Gamma^{\pm}(Y)[R]$ is a component of $\Gamma^{\pm}(\mathcal{Y})$. The *generic matrix* of a signed digraph Γ is the generic matrix of its associated sign pattern, and a signed digraph *requires nonsingularity* if its associated sign pattern requires nonsingularity.

2 Parameters for Minimum Rank and Maximum Nullity of Sign Patterns

In this section we establish relationships between several parameters related to minimum rank and maximum nullity.

2.1 SNS Number

One of the parameters used in the study of minimum rank of digraphs in [1] was the triangle number, and it was shown that if Γ is a ditree then $\mathrm{mr}(\Gamma) = \mathrm{tri}(\Gamma)$. It is easy to give an example of a tree sign pattern $\mathcal{T}$ for which $\mathrm{mr}(\mathcal{T}) > \mathrm{tri}(\mathcal{T})$, e.g., $\mathcal{T} = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$. However, there is a more appropriate generalization of triangle number that retains the property of being equal to minimum rank for tree sign patterns.

The *SNS number* of a sign pattern $\mathcal{Y}$, denoted $\mathrm{SNS}(\mathcal{Y})$, is the maximum size of an SNS subpattern in $\mathcal{Y}$. An $n \times n$ zero-nonzero pattern requires nonsingularity (or equivalently has minimum rank n) if and only if is permutationally similar to a triangle zero-nonzero pattern [1, Prop. 4.6]. Thus the SNS number is a natural generalization of triangle number to sign patterns.

Observation 2.1. *For any sign pattern* $\mathcal{Y}$, $\mathrm{SNS}(\mathcal{Y}) \leq \mathrm{mr}(\mathcal{Y})$.

As is the case with triangle number, the inequality in Observation 2.1 can be strict, as the next example shows.

Example 2.2. Let

$$\mathcal{H} = [\eta_{ij}] = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix}.$$

For the generic matrix $Y_{\mathcal{H}} = [\eta_{ij} y_{ij}]$ of $\mathcal{H}$, it is straightforward to verify that $\det Y_{\mathcal{H}}$ and the determinant of each 3×3 subpattern all have terms of opposite sign. Since $\det Y_{\mathcal{H}}[\{1,2\}] = -y_{11}y_{22} - y_{12}y_{21}$, $\mathrm{SNS}(\mathcal{H}) = 2$. We show that if $H \in \mathcal{Q}(\mathcal{H})$, then $\mathrm{rank}\, H \geq 3$; since $\det \mathcal{Y}_{\mathcal{H}}$ has terms of opposite sign, this implies $\mathrm{mr}(\mathcal{H}) = 3$.

By pre- and post-multiplying by positive diagonal matrices, we may assume H has the form

$$H = \begin{bmatrix} 1 & 1 & h_{13} & h_{14} \\ h_{21} & -1 & 1 & -h_{24} \\ h_{31} & h_{32} & -1 & -1 \\ h_{41} & -h_{42} & -h_{43} & 1 \end{bmatrix}$$

where $h_{ij} > 0$. To show that the rank must be at least 3, we perform Gaussian elimination on the first 2 columns to obtain

$$\begin{bmatrix} 1 & 1 & h_{13} & h_{14} \\ 0 & -1-h_{21} & 1-h_{13}h_{21} & -h_{14}h_{21}-h_{24} \\ 0 & 0 & -\frac{h_{13}h_{32}h_{21}+h_{21}+h_{13}h_{31}+h_{31}-h_{32}+1}{h_{21}+1} & -\frac{h_{14}h_{32}h_{21}+h_{21}+h_{14}h_{31}-h_{24}h_{31}+h_{24}h_{32}+1}{h_{21}+1} \\ 0 & 0 & -\frac{h_{13}h_{41}+h_{41}-h_{13}h_{21}h_{42}+h_{42}+h_{21}h_{43}+h_{43}}{h_{21}+1} & \frac{h_{14}h_{42}h_{21}+h_{21}-h_{14}h_{41}+h_{24}h_{41}+h_{24}h_{42}+1}{h_{21}+1} \end{bmatrix}.$$

If $\operatorname{rank} H = 2$, then the last two rows in this matrix must be 0, but we show this is not possible. If the $3,3$- and $3,4$-entries are 0, then

$$\begin{aligned} h_{13} &= \frac{-h_{21}-h_{31}+h_{32}-1}{h_{31}+h_{21}h_{32}}, \\ h_{14} &= \frac{-h_{21}+h_{24}h_{31}-h_{24}h_{32}-1}{h_{31}+h_{21}h_{32}}. \end{aligned}$$

Since $h_{13} > 0$ and $h_{14} > 0$,

$$h_{32} > h_{21}+h_{31}+1, \tag{1}$$
$$h_{24}h_{31} > h_{21}+h_{24}h_{32}+1. \tag{2}$$

From (1), $h_{32} > h_{31}$, and from (2), $h_{31} > h_{32}$, a contradiction.

Forster's lower bound on minimum rank of an $n \times n$ sign pattern $\mathcal{Y}$ that has no zero entries is

$$\frac{n}{\|A_{\mathcal{Y}}\|}, \tag{3}$$

where $A_{\mathcal{Y}}$ is the matrix obtained from a sign pattern $\mathcal{Y}$ by replacing $+$ by 1 and $-$ by -1 and $\|A\|$ is the spectral norm of A [6]. Note that $\mathcal{H}$ in Example 2.2 is a *Hadamard sign pattern*, i.e., the sign pattern of a Hadamard matrix, and Hadamard sign patterns are what Forster uses to obtain his results on communication complexity. For any $n \times n$ Hadamard matrix H_n, Forster's bound (3) gives $\sqrt{n} \leq \operatorname{mr} \operatorname{sgn}(H_n)$, since $\|H_n\| = \sqrt{n}$. Thus Example 2.2 also shows that the minimum rank of a Hadamard sign pattern can be greater than that given by Forster's lower bound, and suggests further investigation of the minimum ranks of sign patterns of Hadamard matrices.

The next result is used in place of Observation 4.8 in [1].

Lemma 2.3. *Let $\mathcal{Y}'$ be obtained from the $m \times n$ sign pattern $\mathcal{Y}$ by deleting one row. Then* $\mathrm{SNS}(\mathcal{Y}') \geq \mathrm{SNS}(\mathcal{Y}) - 1$.

Proof. Let $\mathcal{Y}'$ be obtained from $\mathcal{Y}$ by deleting row r. Let $R \subseteq \{1,\ldots,m\}, C \subseteq \{1,\ldots,n\}$ be such that $|R| = |C| = \mathrm{SNS}(\mathcal{Y})$ and $\mathcal{Y}[R|C]$ is SNS. If $r \notin R$ then $\mathrm{SNS}(\mathcal{Y}') = \mathrm{SNS}(\mathcal{Y})$. So assume $r \in R$. Let $Y = [\psi_{ij} y_{ij}]$ be the generic matrix for $\mathcal{Y} = [\psi_{ij}]$. Since $\det Y[R|C] \neq 0$, by computing this determinant by the Laplace expansion on row r it is clear that there is an index $c \in C$ such that $\psi_{rc} \neq 0$ and $\det Y[R \setminus \{r\}|C \setminus \{c\}] \neq 0$. Since $\mathcal{Y}[R|C]$ is SNS, all the terms in $\det Y[R \setminus \{r\}|C \setminus \{c\}]$ have the same sign, i.e., $\mathcal{Y}[R \setminus \{r\}|C \setminus \{c\}]$ is SNS. Thus $\mathrm{SNS}(\mathcal{Y}') \geq \mathrm{SNS}(\mathcal{Y}) - 1$. □

2.2 Edit Distance

Edit distance, which was introduced as a parameter of a square zero-nonzero pattern in [1], can be adapted to square sign patterns. Editing row v of the square sign pattern $\mathcal{Y}$ is equivalent to editing the out-neighborhood of vertex v in $\Gamma^{\pm}(\mathcal{Y})$.

Definition 2.4. Let $\mathcal{Y}$ be a square sign pattern. The *(row) edit distance to nonsingularity*, $\mathrm{ED}(\mathcal{Y})$, of $\mathcal{Y}$ is the minimum number of rows that must be changed to obtain an SNS pattern.

The proof of the next result can be adapted from the proof of [1, Theorem 4.9], with SNS number replacing triangle number; see also [9].

Theorem 2.5. *For any $n \times n$ sign pattern $\mathcal{Y}$, $\mathrm{SNS}(\mathcal{Y}) + \mathrm{ED}(\mathcal{Y}) = n$.*

Corollary 2.6. *For any sign pattern $\mathcal{Y}$, $\mathrm{M}(\mathcal{Y}) \leq \mathrm{ED}(\mathcal{Y})$.*

2.3 Path Cover Number

We extend the definition of path cover number given in [1] to square sign patterns. Following the definition in that paper, paths are not required to be induced, whereas in many papers studying (symmetric) minimum rank, paths are required to be induced (see [5] and the references therein). However, the distinction is irrelevant for tree sign patterns, to which the results below will be applied in Section 3.

Definition 2.7. The path cover number $\mathcal{P}(\mathcal{Y})$ of $\mathcal{Y}$ is the minimum number of vertex-disjoint paths whose deletion from $\Gamma^{\pm}(\mathcal{Y})$ leaves a signed digraph that requires nonsingularity (or the empty set), i.e., the deletion of the rows and columns corresponding to the vertices of the paths leaves an SNS sign pattern (or the empty set).

In [1], it is shown that $\mathcal{P}(\Gamma) \leq \mathrm{ED}(\Gamma)$ for a digraph Γ. The proof given there uses the zero forcing number, which we have not been able to adapt to sign patterns in a useful way. In the next theorem we give a (different) direct proof of the analogous inequality for sign patterns.

Theorem 2.8. *For any square sign pattern $\mathcal{Y}$, $\mathcal{P}(\mathcal{Y}) \leq \mathrm{ED}(\mathcal{Y})$.*

Proof. Let $\widetilde{\mathcal{Y}}$ be an SNS sign pattern obtained from $\mathcal{Y}$ by editing rows $r_1, \ldots, r_e$, where $e = \mathrm{ED}(\mathcal{Y})$, let $\widetilde{\Gamma} = \Gamma^{\pm}(\widetilde{\mathcal{Y}})$, and let $\Gamma = \Gamma^{\pm}(\mathcal{Y})$. Since $\widetilde{\mathcal{Y}}$ is an SNS pattern, all terms in the determinant of the generic matrix $\widetilde{Y}$ of $\widetilde{\mathcal{Y}}$ have the same sign. Select one nonzero term t in

$\det \widetilde{Y}$. Note that t is a generalized cycle product. Let $C_i, i = 1, \ldots, f \le e$ be the cycles in $\widetilde{\Gamma}$ associated with the simple cycle products in t that contain entries from rows $r_1, \ldots, r_e$. If C_i contains $r_{k_1}, \ldots, r_{k_{s_i}}$ (in that order on the cycle), then denote the cycle vertex immediately following r_{k_j} by $u_{k_{j+1}}$ where $s_i + 1$ is interpreted as 1. If there are no other vertices between the vertices r_{k_j} and $r_{k_{j+1t}}$, then $u_{k_{j+1}} = r_{k_{j+1}}$. All the vertices of cycle C_i can be deleted from $\mathcal{Y}$ by deleting the s_i paths in Γ consisting of the vertices and cycle arcs from u_{i_j} to $r_{i_j}, j = 1, \ldots, s_i$. Note that the arcs involved in these paths are all in Γ, because the only cycle arcs of $\widetilde{\Gamma}$ that may not exist in Γ are the arcs $(r_{k_j}, u_{k_{j+1}})$, which are not used in these paths. Let $V_C = \cup_{i=1}^{f} V_{C_i}$. We claim that $\widetilde{\mathcal{Y}} - V_C = \mathcal{Y} - V_C$ is an SNS sign pattern or the empty set; one this is established, it is clear that $\mathcal{P}(\mathcal{Y}) \le \mathrm{ED}(\mathcal{Y})$.

Assume $\{1, \ldots, n\} \setminus V_C$ is nonempty. Since we have removed vertices of complete cycles in the generalized cycle product t, t can be factored as $t = t_1 t_2$, where all of the indices in cycle products in t_1 are in V_C and all of the indices in cycle products in t_2 are in $\{1, \ldots, n\} \setminus V_C$. Thus t_2 is a nonzero term in $\det Y'$ where Y' is the generic matrix of $\widetilde{\mathcal{Y}} - V_C$. Suppose that there is a term t_3 in $\det Y'$ that has the opposite sign from t_2. Then $t_1 t_3$ would be a term of opposite sign from $t = t_1 t_2$ in $\det \widetilde{Y}$, contradicting the fact that $\widetilde{\mathcal{Y}}$ is an SNS sign pattern. Thus all the terms in $\det Y'$ have the same sign and $\widetilde{\mathcal{Y}} - V_C = \mathcal{Y} - V_C$ is an SNS sign pattern. □

It is easy to find an example of a sign pattern that has path cover number strictly less than edit distance and maximum nullity.

Example 2.9. (cf. [1, Example 4.21]) Let $[+]_n$ denote the $n \times n$ sign pattern consisting entirely of positive entries. If $n \ge 3$, then

$$\mathcal{P}([+]_n) = 1 < n - 1 = \mathrm{M}([+]_n) = \mathrm{ED}([+]_n).$$

3 Tree sign patterns

All the proofs of the results in this section are omitted because they can be adapted from the proofs of the corresponding results in Section 5 of [1], using results in Section 2 of this paper to replace those in Section 4 of [1]; the proofs are also available in [9].

As noted in Example 2.9, it is possible to have $\mathcal{P}(\mathcal{Y}) < \mathrm{ED}(\mathcal{Y})$ for a sign pattern $\mathcal{Y}$, but the next theorem shows that this is not possible for tree sign patterns.

Theorem 3.1. (cf. [1, Theorem 5.1]) *For every tree sign pattern $\mathcal{T}$, $\mathrm{ED}(\mathcal{T}) \le \mathcal{P}(\mathcal{T})$.*

Using Theorems 2.8 and 3.1, we have the following corollary.

Corollary 3.2. *For every tree sign pattern $\mathcal{T}$,*

$$\mathcal{P}(\mathcal{T}) = \mathrm{ED}(\mathcal{T}).$$

Lemma 3.3. (cf. [1, Lemma 5.5]) *Let $\mathcal{T}$ be a combinatorially symmetric $n \times n$ tree sign pattern, let $v \in \{1, \ldots, n\}$, and let $S \subseteq \{1, \ldots, n\}$ such that*

1. *$\mathcal{T}[S]$ is a component of $\mathcal{T} - v$,*

2. $\mathcal{T}[S]$ *allows singularity, and*

3. *if* $x \in S$*, then* $\mathcal{T} - x$ *has at most one component that is a subpattern of* $\mathcal{T}[S]$ *and allows singularity.*

Then there is a path P *in* $\Gamma^{\pm}(\mathcal{T})$ *from* v *to a vertex* $u \in S$ *such that every component of* $\mathcal{T} - V_P$ *that is a subpattern of* $\mathcal{T}[S]$ *is SNS.*

Theorem 3.4. (cf. [1, Theorem 5.6]) *If* $\mathcal{T}$ *is a combinatorially symmetric tree sign pattern, then*

$$\mathrm{M}(\mathcal{T}) = \mathcal{P}(\mathcal{T}) = \mathrm{ED}(\mathcal{T}) \qquad \textit{and} \qquad \mathrm{mr}(\mathcal{T}) = \mathrm{SNS}(\mathcal{T}).$$

Lemma 3.5. (cf. [1, Lemma 5.7]) *Let* $\mathcal{Y}$ *be a sign pattern of the form*

$$\mathcal{Y} = \begin{bmatrix} \mathcal{X} & O \\ \mathcal{U} & \mathcal{W} \end{bmatrix}.$$

where $\mathcal{U}$ *is* $k \times m$*,* $\mathcal{U}$ *has exactly one nonzero entry, and that entry is in the* $1, m$*-position of* $\mathcal{U}$ *(which is the same as the* $k+1, m$*-position of* $\mathcal{Y}$*). Let* $\widetilde{\mathcal{X}}$ *be obtained from* $\mathcal{X}$ *by replacing the last column of* $\mathcal{X}$ *by* 0*s and* $\widetilde{\mathcal{W}}$ *be obtained from* $\mathcal{W}$ *by replacing the first row of* $\mathcal{W}$ *by* 0*s. If* $\mathrm{mr}(\mathcal{X}) = \mathrm{SNS}(\mathcal{X})$*,* $\mathrm{mr}(\mathcal{W}) = \mathrm{SNS}(\mathcal{W})$*,* $\mathrm{mr}(\widetilde{\mathcal{X}}) = \mathrm{SNS}(\widetilde{\mathcal{X}})$*,* $\mathrm{mr}(\widetilde{\mathcal{W}}) = \mathrm{SNS}(\widetilde{\mathcal{W}})$*, then* $\mathrm{mr}(\mathcal{Y}) = \mathrm{SNS}(\mathcal{Y})$*.*

Theorem 3.6. (cf. [1, Theorem 5.8]) *If* $\mathcal{T}$ *is a tree sign pattern, then*

$$\mathrm{M}(\mathcal{T}) = \mathcal{P}(\mathcal{T}) = \mathrm{ED}(\mathcal{T}) \qquad \textit{and} \qquad \mathrm{mr}(\mathcal{T}) = \mathrm{SNS}(\mathcal{T}).$$

Note that whereas the parameters $\mathrm{mr}(\mathcal{T})$ and $\mathrm{M}(\mathcal{T})$ involve optimizing over an infinite set of matrices, the computation of $\mathrm{SNS}(\mathcal{T})$ or $\mathcal{P}(\mathcal{T})$ involves only a finite (but possibly very large) number of subsets of vertices. Thus Theorem 3.6 allows (at least in theory) the computation of $\mathrm{mr}(\mathcal{T})$ and $\mathrm{M}(\mathcal{T})$ for a tree sign pattern $\mathcal{T}$. For small tree sign patterns, $\mathcal{P}(\mathcal{T})$ can be computed effectively by hand. A program for the computation of $\mathrm{SNS}(\mathcal{T})$ is available [7] using the free open-source computer mathematics software system *Sage* [13].

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