

Minimum Rank of Symmetric Matrices of a Graph

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Minimum rank

All matrices discussed are symmetric. Until further notice, matrices are real. All graphs discussed are simple.

The family of symmetric matrices described by a graph is

$$\mathcal{S}(G) = \{A : A^T = A \text{ \& for } i \neq j, a_{ij} \neq 0 \Leftrightarrow i \sim j\}.$$

The minimum rank of graph G :

$$\text{mr}(G) = \min_{A \in \mathcal{S}(G)} \text{rank } A.$$

Problem Determine the minimum rank of a graph G .

Minimum rank and maximum nullity

The maximum nullity:

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{null}(A)\}.$$

- $M(G) + \text{mr}(G) = |G|.$
- $M(G)$ is the maximum multiplicity of an eigenvalue of a matrix in $\mathcal{S}(G).$

Minimum Rank Problem for Trees

Let T be a tree.

- $\Delta(T) = \max\{p_Q - |Q| : Q \subseteq V(T)$
and $T - Q$ consists of p_Q disjoint paths}.
- $P(T) =$ minimum number of paths that cover vertices of T .

Theorem

[Johnson, Leal Duarte 99]

$$|T| - \text{mr}(T) = M(T) = \Delta(T) = P(T)$$

There are good algorithms for computing $\Delta(T)$. Methods for computing $\text{mr}(T)$ appeared earlier but less explicitly in [Nylen 96].

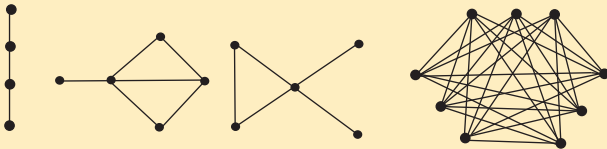
Extreme minimum rank

If G is connected,

- $\text{mr}(G) = 0$ if and only if G is single vertex.
- $\text{mr}(G) = 1$ if and only if $G = K_n$, $n \geq 2$.

Theorem

[Barrett, van der Holst, Loewy 04] *For a connected graph G , $\text{mr}(G) \leq 2$ if and only if G does not contain as an induced subgraph any of*

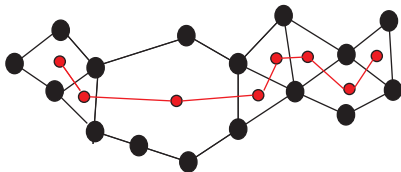


Theorem

[Fiedler 69] $M(G) = 1$ if and only if G is a path.
 $\text{mr}(G) = |G| - 1$ if and only if G is a path.

A connected graph is **2-connected** if at least two vertices must be deleted to disconnect it.

A **2-connected partial 2-path** G is a “path” built of cycles by edge identification. A 2-connected partial 2-path is also called a **Linear Singly Edge Articulated Cycle (LSEAC) graph**.



A 2-connected partial 2-path.

Theorem

[H, van der Holst 07], [Johnson, Loewy, Smith] *For a 2-connected graph G , $M(G) = 2$ if and only if G is partial 2-path.*

Example: Find the minimum rank of G :



$$\text{mr}(G) = 16 - 2 = 14.$$

Theorem

[Barioli, Fallat, H 04], [Hsieh] (cut-vertex reduction)

Let v be a cut-vertex of G . For $i = 1, \dots, h$, let $W_i \subseteq V(G)$ be the vertices of the i th component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then

$$\text{mr}(G) = \min \left\{ \sum_1^h \text{mr}(G_i - v) + 2, \sum_1^h \text{mr}(G_i) \right\}.$$

Colin de Verdière-type parameter ξ

Definition X fully annihilates A if

1. $AX = 0$;
2. $A \circ X = 0$, i.e. X has 0 where A has nonzero;
3. $I_n \circ X = 0$, i.e., all diagonal elements of X are 0.

Definition The matrix A has the **Strong Arnold Hypothesis (SAH)** if the zero matrix is the only symmetric matrix that fully annihilates A . SAH is equivalent to transverse intersection of the constant rank manifold and the constant pattern manifold.

- Colin de Verdière defined parameters μ , ν are maximum nullity of matrices $A \in \mathcal{S}(G)$ satisfying SAH and other conditions.
- H is a **minor** of G if H is obtained from G by a sequence of edge deletions, isolated vertex deletions and edge contractions (in any order).
- μ , ν are minor monotone.
- Minor monotonicity is very useful and closely related to SAH.
- The “other conditions” on μ , ν are irrelevant for minimum rank.

The new parameter ξ

$$\xi(G) = \max\{\text{null}(A) : A \in \mathcal{S}(G), A \text{ has SAH}\}$$

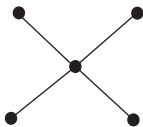
$$\xi(G) \leq M(G)$$

Theorem

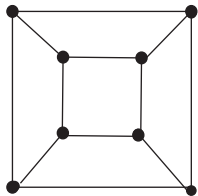
[Barioli, Fallat, H 05]

If H is a minor of G , then $\xi(H) \leq \xi(G)$.

Example: $\xi(K_{1,4}) = 2 < 3 = M(K_{1,4})$.



$K_{1,4}$



Q_3

Example: [van der Holst]
 $\xi(Q_3) = 4 = M(Q_3)$.

AIM Workshop and Minimum Rank

Many new results were obtained at

American Institute of Mathematics Workshop
Spectra of families of matrices described by graphs, digraphs, and
sign patterns
October 23 to October 27, 2006
and subsequently.

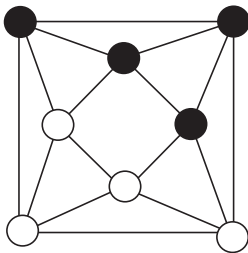
<http://aimath.org/pastworkshops/matrixspectrum.html>

This report is based on the work of the following people:
Francesco Barioli, Wayne Barrett, Avi Berman, Richard Brualdi,
Steven Butler, Sebastian Cioaba, Dragoš Cvetković, Jane Day,
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Sciriha, Bryan Shader, Wasin So, Dragan Stevanović, Pauline van
den Driessche, Hein van der Holst, Kevin Vander Meulen, Amy
Wangsness, Amy Yielding.

Color change rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G , and exactly one neighbor v of u is white, then change the color of v to black.

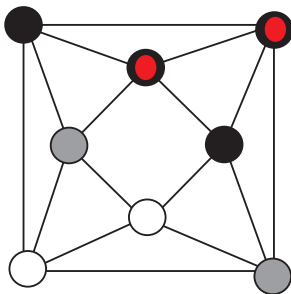
Example H_8



Color change rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G , and exactly one neighbor v of u is white, then change the color of v to black.

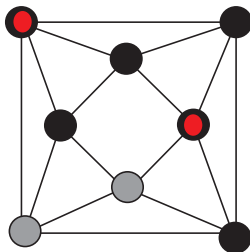
Example (H_8 continued)



Color change rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G , and exactly one neighbor v of u is white, then change the color of v to black.

Example (H_8 continued)



Zero-Forcing Sets

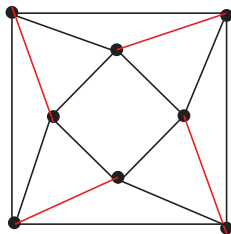
- Given a coloring of G , the **derived coloring** is the result of applying the color-change rule until no more changes result.
- A **zero forcing set** for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

Theorem

[AIM] For any graph G , $M(G) \leq Z(G)$.

Example $M(H_8) \leq Z(H_8) \leq 4$ and so $\text{mr}(H_8) \geq 4$.

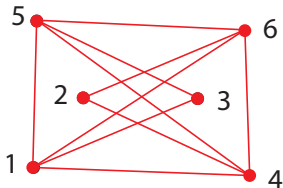
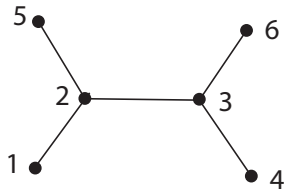
In fact $\text{mr}(H_8) = 4$. This is established by showing $\xi(Q_3) = 4$ and noting Q_3 is a minor.



Graph Complements

If $G = (V, E)$ is a graph, the **complement** is $\overline{G} = (V, \overline{E})$.

Example: A graph and its complement



Orthogonal Representations

Orthogonal representations have been used to compute minimum rank of graph complements.

- $\varphi : V \rightarrow \mathbb{R}^d$ such that $\varphi(u)$ and $\varphi(v)$ are orthogonal if and only if u and v are nonadjacent vertices is an **orthogonal representation** of G in \mathbb{R}^d .
- If $\varphi : V \rightarrow \mathbb{R}^d$ is an orthogonal representation of G , then $\text{mr}(G) \leq d$ (since the Gram matrix of the vectors $\varphi(u)$ is a positive semi-definite matrix in $\mathcal{S}(G)$)

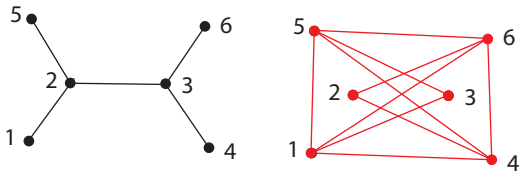
Theorem

[AIM] $\text{mr}(\overline{T}) \leq 3$ for T a tree.

Proof:

- Let L_u be the plane orthogonal to $\varphi(u)$.
- Construct T by adding one leaf at a time. Simultaneously construct an orthogonal representation $\varphi : V(\overline{T}) \rightarrow \mathbb{R}^3$ such that $\varphi(v)$ and $\varphi(w)$ are linearly independent for distinct vertices.
- Let w be the vertex adjacent to leaf v added T at this stage. Choose a vector \mathbf{x} in L_w which is not in L_u for all $u \in V(\overline{T} - v - w)$. and not a multiple of $\varphi(u)$ for $u \in V(\overline{T} - v - w)$.
- Define $\varphi(v) = \mathbf{x}$.

Example: A tree T and its complement \overline{T} .



Orthogonal representation of \overline{T} : $B = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 & 1 & 2 \\ 1 & 0 & 2 & -3 & 3 & -3 \end{bmatrix}$

$$A = B^T B = \begin{bmatrix} 3 & 0 & 4 & 3 & 5 & 3 \\ 0 & 2 & 0 & -2 & 0 & 2 \\ 4 & 0 & 6 & 0 & 8 & 0 \\ 3 & -2 & 0 & 29 & -3 & 25 \\ 5 & 0 & 8 & -3 & 11 & -3 \\ 3 & 2 & 0 & 25 & -3 & 29 \end{bmatrix} \in \mathcal{S}(\overline{T})$$

A **unicyclic graph** is a graph with exactly one cycle.

A **2-tree** is obtained from K_3 by adding one vertex at a time adjacent to a pair of already adjacent vertices.

Orthogonal representations can be used to establish bounds on the complements of these graphs.

Theorem

- $\text{mr}(\overline{T}) \leq 3$ for T a tree [AIM]
- $\text{mr}(\overline{G}) \leq 4$ for G a unicyclic graph [H]
- $\text{mr}(\overline{G}) \leq 4$ for G a 2-tree [H]
- $\text{mr}(\overline{C_n}) = 3$ ($n \geq 5$) where C_n is an n -cycle [AIM]

What can be said about the relationship between $\text{mr}(G)$ and $\text{mr}(\overline{G})$?

Trivially if $|G| \geq 2$,

$$1 \leq \text{mr}(G) + \text{mr}(\overline{G}).$$

For all graphs of order ≤ 6 and all examples where both minimum ranks are known

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2.$$

Theorem

$$\text{mr}(G) + \text{mr}(\overline{G}) \leq |G| + 2$$

for the following graphs

- K_n (trivial)
- C_n [AIM]
- trees [AIM]
- unicyclic graphs [H]
- 2-trees [H]

Minimum Rank over Other Fields

Let F be a field.

The family of symmetric matrices over F described by a graph is

$$\mathcal{S}^F(G) = \{A : A \in F^{n \times n}, A^T = A, \text{ and } \mathcal{G}(A) = G\}.$$

The minimum rank of graph G over F :

$$\text{mr}^F(G) = \min_{A \in \mathcal{S}^F(G)} \text{rank } A.$$

Problem Determine the minimum rank of a graph G over F .

Let F be a field.

- Study of $\text{mr}^F(G)$ initiated by Barrett, Loewy and van der Holst.
- Minimum rank depends on field.
- Until AIM workshop, only examples of field dependence required fields to differ in characteristic or order.

Theorem

[Berman, Friedland, H, Rothblum, Shader]

Examples of graphs H and K such that

$$\text{mr}^{\mathbb{C}}(H) < \text{mr}^{\mathbb{R}}(H) \text{ and } \text{mr}^{\mathbb{R}}(K) < \text{mr}^{\mathbb{Q}}(K).$$

- These examples were constructed using matroids.
- K gives a counterexample to a conjecture of Arav, Hall, Koyucu, Li and Rao that the minimum rank of any sign pattern can be realized by a rational matrix.

AIM Minimum Rank Graph Catalog

To summarize what is known and facilitate making conjectures, two catalogs of graphs and their minimum ranks were developed following the AIM workshop.

- Catalog for families of graphs
<http://aimath.org/pastworkshops/catalog2.html>
 - Previously known results such as trees
 - Numerous results obtained by the AIM workshop using zero forcing sets and other techniques.
- Catalog for small graphs
<http://aimath.org/pastworkshops/catalog1.html>
 - All graphs of order 6 or less
 - Selected larger graphs



Thank you!

