Variants on the minimum rank problem: A survey II*

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Abstract

The minimum rank problem for a (simple) graph G is to determine the smallest possible rank over all real symmetric matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. This paper surveys the many developments on the (standard) minimum rank problem and its variants since the survey paper [36]. In particular, positive semidefinite minimum rank, zero forcing parameters, and minimum rank problems for patterns are discussed.

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AMS subject classifications. 05C50, 15A03, 15B57, 15B35, 15A18

1 Introduction

Since our survey paper [36] the volume of work, advances, and interesting open problems on many different aspects of the minimum rank of graphs has continued to expand. Furthermore, since the 2006 AIM workshop that featured graphs and minimum rank, there have been numerous special sessions, minisymposia, and a BIRS workshop emphasizing the topic of minimum rank of graphs. Consequently, we felt it was timely to produce an updated survey covering more recent topics and advances on the minimum rank of graphs which is meant to serve as a sequel to the original survey paper [36].

Since this work is follow up reporting, we will not repeat all of the necessary notation or terminology that was presented in [36], so please consult [36] if relevant terms or notation are not spelled out here. However, we will carefully define key terms and notation used within.

In general the minimum rank of a graph is simply the smallest rank over a collection of matrices that are in some way associated with a given graph G. As was outlined in [36], this simple question has it roots in many different topics in combinatorics and has been a concern for many researchers over the years. Recently, connections have been found between the related graph parameter zero forcing number and control of quantum systems (see Section 4), and between the minimum rank of sign patterns and communication complexity (see Section 5.3). As mentioned above, minimum rank problems are a hot topic currently and has seen a tremendous boom in results and applications over the past 10 years (see references).

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As usual, a *graph* is a pair $G = (V, E)$, where V is the set of vertices (typically $\{1, \ldots, n\}$ or a subset thereof) and E is the set of edges (an edge is a two-element subset of vertices). A *general graph* allows multiple edges and/or loops. Every graph or general graph considered here is finite (finite number of vertices and finite number of edges) and has a nonempty vertex set. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $B \in S_n(\mathbb{R})$, the *graph* of B, denoted $\mathcal{G}(B)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} |\ b_{ij} \neq 0 \text{ and } i \neq j\}$. Note that the diagonal of B is ignored in determining $\mathcal{G}(B)$. In addition, we let $\mathcal{S}(G) = \{B \in S_n(\mathbb{R}) : \mathcal{G}(B) = G\}$. Observe that for a given graph G, the classical matrices such as the *adjacency matrix* of G, the *Laplacian matrix* of G, and the *signless Laplacian matrix* of G all lie in $\mathcal{S}(G)$.

Example 1.1. For the matrix $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 3.1 & -1.5 & 2 \\ 0 & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$, $\mathcal{G}(B)$ is shown in Figure 1.

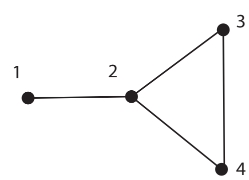

Figure 1: The graph $\mathcal{G}(B)$ for B in Example 1.1

Then the *minimum rank* of a graph G of order n is defined to be

$$\operatorname{mr}(G) = \min\{\operatorname{rank} B : \ B \in S_n(\mathbb{R}) \text{ and } \mathcal{G}(B) = G\}.$$

The problem of determining $\operatorname{mr}(G)$ is often referred to as the *standard minimum rank problem*. The *maximum multiplicity* of G is given as

$$M(G) = \max\{\operatorname{mult}_B(\lambda) : \ \lambda \in \mathbb{R}, B \in S_n(\mathbb{R}) \text{ and } \mathcal{G}(B) = G\}.$$

Translating by a scalar matrix if necessary, it is clear that the maximum multiplicity of any eigenvalue is the same as maximum multiplicity of the eigenvalue 0. Thus maximum multiplicity is sometimes called *maximum nullity* or even *maximum corank*.

The following results are well-known, straightforward, and were presented in [36].

1. $M(G) + \operatorname{mr}(G) = |G|$.
2. $\operatorname{mr}(G) \leq |G| - 1$.
3. $\operatorname{mr}(P_n) = n - 1$, ($P_n$ denotes the path on n vertices).
4. $\operatorname{mr}(K_n) = 1$, and if G is connected, $\operatorname{mr}(G) = 1$ implies $G = K_{|G|}$, that is, G is the complete graph on $|G|$ vertices.

Example 1.2. Let G be the graph in Figure 1 and let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Since $\mathcal{G}(A) = G$, $G \neq K_4$, and $\operatorname{rank} A = 2$, it follows that $\operatorname{mr}(G) = 2$.

For a more detailed introduction to this topic and a broad list of fundamental results on the minimum rank of graphs, please consult [36]. This present survey is divided into five sections. The next section represents an update on recent advances and directions regarding the standard minimum rank problem (that is, on the parameter $\mathrm{mr}(G)$). The third section of this survey discusses a variant of minimum rank restricted to the subset of $\mathcal{S}(G)$ consisting of all the positive semidefinite matrices and the corresponding positive semidefinite minimum rank of a graph. The fourth section reviews a recent combinatorial parameter, known as the zero forcing number, and outlines its history and various other types of zero forcing parameters, along with their connection to the maximum nullity of a graph. The final two sections are devoted to problems that are related to the minimum rank of a graph but are different in concept. For example, in Section 5, we consider the ranks of matrices associated with directed graphs and sign patterns, and in Section 6, we discuss related problems for (simple) graphs, such as matrices over fields other than the real numbers, the inverse inertia problem, and minimum skew rank.

2 Update on the standard minimum rank problem

Since the first survey [36] four years ago, about 20 papers have appeared with results about the standard minimum rank problem, i.e., the problem of determining the minimum rank $\mathrm{mr}(G)$ of a simple graph describing the off-diagonal nonzero pattern of real symmetric matrices. A variety of results have appeared computing minimum rank for specific families of graphs, e.g., graphs of order at most 7 [32], equivalence class graphs [37], ciclos and estrellas [3]. Huang, Chang, and Yeh study various families having maximum nullity equal to zero forcing number (see Section 4), including block-clique graphs and unit-interval graphs [49]. Barioli, Fallat, and Smith characterized graphs having minimum rank equal to diameter [13]. Barrett et al determined the effect on minimum rank of certain graph operations such as edge subdivision [14]. Hogben and Shader studied the effect on maximum nullity of requiring null vectors to be generic [45]. Recall that a path cover of a graph G is a set of vertex disjoint induced paths that cover all the vertices of G, and the path cover number $\mathrm{P}(G)$ is the minimum number of paths in a path cover of G. Sinkovic showed that for an outerplanar graph G, $\mathrm{M}(G)$ is bounded above by the path cover number $\mathrm{P}(G)$ [67].

Section 2.1 below describes computer programs that are now available for computing the minimum rank of small graphs. Section 2.2 describes work on determining the average minimum rank over all (labeled) graphs of a fixed order. Some of the progress on the standard minimum rank problem is discussed in other sections of this article. The zero forcing number, whose terminology was developed at the AIM workshop [4], has played a role in much of the recent progress on minimum rank. This parameter, its extensions, and applications to physics are described in Section 4. The graph complement conjecture (GCC) was posed as a question at the AIM workshop. Although still unproved, progress has been made on GCC, and it is now believed that stronger positive semidefinite versions of the conjecture are true. Thus work on GCC is discussed in Section 3 rather than in this section. The delta conjecture was also discussed at the AIM workshop (and a stronger positive semidefinite version was conjectured by Maehara in 1987 [57]); the delta conjecture is discussed in Section 3.

2.1 Software for minimum rank, maximum nullity, and zero forcing number

Since 2008 several programs have been written in the computer mathematics system *Sage* to compute various known bounds on minimum rank and maximum nullity for a given graph. For a small graph (e.g., order at most 10) the upper and lower bound are often equal, thereby providing the minimum rank. These were originally published in [33]. Subsequently, improvements have been made, primarily to the computation of zero forcing parameters, and the 2010 state of the art version is available at [25]. These programs enabled experimentation that led to the discovery of the estrella $S_5(K_4)$, a 3-connected planar graph that has the property $\mathrm{M}(S_5(K_4)^d) \neq \mathrm{M}(S_5(K_4))$ [3] (here G^d is the dual of G).

2.2 Average minimum rank

Most of the graph families for which minimum rank has been computed are *sparse*, meaning that the number of edges is much less than the maximum possible number of edges $\binom{n}{2} \approx \frac{1}{2}n^2$, and most are structured and exhibit symmetry. However, the random graph $G(n, \frac{1}{2})$ for which it is equally likely that each edge is present or absent (the probability of each edge is $\frac{1}{2}$) is expected to have $\frac{n(n-1)}{4}$ edges. Thus the graphs for which minimum rank has been computed tend to present a somewhat atypical picture. Hall, Hogben, Martin, and Shader [42] obtained bounds on the average value of minimum rank (over all labeled graphs of a fixed order).

Formally, the *average minimum rank* of graphs of order n is the sum over all labeled graphs of order n of the minimum ranks of the graphs, divided by the number of (labeled) graphs of order n. That is,

$$\mathrm{amr}(n) = \frac{\sum_{|G|=n} \mathrm{mr}(G)}{2^{\binom{n}{2}}}.$$

The average minimum rank is equal to the expected value of the minimum rank of $G(n, \frac{1}{2})$, denoted by $E[\mathrm{mr}(G(n, 1/2))]$. The main results on average minimum rank are

Theorem 2.1. *For n sufficiently large,*

1. $0.146907n < \mathrm{amr}(n) < 0.5n + \sqrt{7n \ln n}$, *and*

2. $|\mathrm{mr}(G(n, 1/2)) - \mathrm{amr}(n)| < \sqrt{n \ln \ln n}$ *with probability approaching* 1 *as* $n \to \infty$.

The results in [42] are somewhat more general. Asymptotic bounds are obtained for $E[\mathrm{mr}(G(n, p))]$, the expected value of the minimum rank of $G(n, p)$, where p is the probability that an edge is present, and for the expected value of the Colin de Verdière type parameter ξ.

3 Positive semidefinite minimum rank

Associating mathematical objects to the vertices of a graph has long been a useful tool in graph theory. This technique also has roots in certain minimum rank problems.

A standard example is assigning vectors to the vertices of a graph in such a way that orthogonality corresponds to non-adjacency. That is, for any pair of vertices u, v in G, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ assigned to u and v are orthogonal if and only if $\{u, v\} \notin E$.

For example, if G is the graph from Figure 1, then assigning the standard basis vector $\mathbf{e}_1$ from $\mathbb{R}^2$ to vertex 1, $\mathbf{e}_2 \in \mathbb{R}^2$ to vertices 3 and 4, and $\mathbf{e}_1 + \mathbf{e}_2$ to vertex 2, is a labeling of the vertices that respects the condition of having nonadjacent vertices assigned to orthogonal vectors. Also observe that if

$$B = \left[\begin{array}{cccc} \mathbf{e}_1, & \mathbf{e}_1 + \mathbf{e}_2, & \mathbf{e}_2, & \mathbf{e}_2 \end{array} \right],$$

then B is a 2×4 real matrix such that $B^T B$ is a positive semidefinite matrix in $\mathcal{S}(G)$. Moreover, the rank of $B^T B$ is two. Hence the minimum rank among all positive semidefinite matrices in $\mathcal{S}(G)$ is at most two (in fact, it is exactly two in this instance and $B^T B$ is equal to the matrix A in Example 1.2).

For any graph G of order n, we let $\mathcal{S}_+(G)$ denote the subset of $\mathcal{S}(G)$ consisting of all real positive semidefinite matrices. Further, we let

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}_+(G)\},$$

and

$$\mathrm{M}_+(G) = \max\{\mathrm{null}\, A : A \in \mathcal{S}_+(G)\}.$$

The parameter $\mathrm{mr}_+(G)$ is called the *(real) minimum positive semidefinite rank of* G, while $\mathrm{M}_+(G)$ is called the *maximum positive semidefinite nullity of* G. As with the case of standard minimum rank, it is clear that for any graph G

$$\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|.$$

Now, following the example above, if G is a graph and for each vertex $i \in V$ we assign the vector $\mathbf{v}_i \in \mathbb{R}^d$ such that $\mathbf{v}_i^T \mathbf{v}_j = 0$ if and only if $\{i,j\} \notin E$, then the matrix $B^T B$, where $B = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$ is in $\mathcal{S}_+(G)$ with rank equal to k. Such a vector representation is called an orthogonal vector representation (see also [57, 58] where the representation above is known as a *faithful* orthogonal vector representation). Orthogonal vector representations also arise in the works [22, 39, 62] mostly over the complex field, but the concept is analogous. Orthogonal vector representations (of the non-faithful variety) also appear in connection with the Lovász ϑ function and related versions of certain sandwich type theorems (see, for example, [36] and the relevant references within). It follows easily that $\mathrm{mr}_+(G)$ coincides with the smallest d such that G admits an orthogonal vector representation with vectors from $\mathbb{R}^d$.

As noted above, it is also of interest to investigate the smallest d such that the graph G admits an orthogonal vector representation with vectors lying in $\mathbb{C}^d$ instead of restricting to the real case. The smallest such d will be denoted by $\mathrm{mr}_+^{\mathbb{C}}(G)$ and it is not difficult to observe that

$$\mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{H}^+(G)\},$$

where $\mathcal{H}^+(G)$ is the subset of positive semidefinite matrices among all complex Hermitian matrices A such that $\mathcal{G}(A) = G$. This term has been well-studied just like its real counterpart and in the papers [22, 39, 62] we note that $\mathrm{mr}_+^{\mathbb{C}}(G)$ is denoted by the symbol $\mathrm{msr}(G)$. It is very important to observe that changing fields from $\mathbb{R}$ to $\mathbb{C}$ does result in a different parameter as noted in [7].

In many ways, it does appear that the parameters $\mathrm{mr}_+(G)$ and $\mathrm{mr}_+^{\mathbb{C}}(G)$ may be more natural graph-type parameters when compared to other notions of minimum rank. This opinion may be defended by the simplicity of many results about minimum positive semidefinite rank and its connections to graph theory.

For example, it is known that the minimum semidefinite rank of any tree is precisely the order of the tree less one, which is as large as the minimum semidefinite rank can be in general (see, for example, [48] or [22]).

In the context of certain graph operations, the minimum semidefinite rank behaves rather nicely. For example, in the case when G has a cut vertex the minimum semidefinite rank of G can be computed by summing the minimum semidefinite ranks of smaller graphs (see [22] for a proof over the complex numbers, although a similar argument will work over the reals, see also [48]). We note here that the formula below can easily be used with a simple induction argument to verify that the minimum semidefinite rank of trees is precisely the order of the tree less one.

Theorem 3.1. *Suppose G has a cut-vertex v. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the ith component of $G - v$ and let G_i be the subgraph induced by $\{v\} \cup W_i$. Then*

$$\mathrm{mr}_+(G) = \sum_1^h \mathrm{mr}_+(G_i).$$

An analogous cut-vertex reduction formula for minimum rank for a graph was obtained earlier by various authors and is presented in [36]. However that formula is more convoluted and depends on the notion of the rank-spread of a vertex. Recall that the rank-spread of G at vertex v is defined to be $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$. In the positive semidefinite case it is not difficult to observe that the rank spread of a vertex v is bounded between

$$0 \leq \mathrm{mr}_+(G) - \mathrm{mr}_+(G - v) \leq \deg(v),$$

where $\deg(v)$ is the degree of the vertex v. The fact that the rank spread in the positive semidefinite case can be larger than 2 seems to simplify calculations in the case of cut vertex reduction.

In the case of the join of two graphs a similar simplification occurs. Recall that the *join* $G \vee G'$ of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the union of $G \cup G'$ and the complete bipartite graph with with vertex set $V \cup V'$ and partition $\{V, V'\}$.

The following fact was proved in [39] over the complex numbers and in [9] over the reals.

Theorem 3.2. *If G and H are two graphs, then*

$$\mathrm{mr}_+(G \vee H) = \max\{\mathrm{mr}_+(G \vee K_1), \mathrm{mr}_+(H \vee K_1)\}, \tag{1}$$

where K_1 is the complete graph on a single vertex.

Observe that if G and H do not contain any isolated vertices, then we have

$$\mathrm{mr}_+(G \vee H) = \max\{\mathrm{mr}_+(G), \mathrm{mr}_+(H)\}.$$

For standard minimum rank it is well-known that the equations above need not hold in general, and, in fact, $\mathrm{mr}(G \vee H)$ only behaves as in (1) for the special case of graphs that are among the so-called inertia-balanced and not anomalous (see [36]). For example, if G and H are both trees or are both decomposable graphs, then (1) is valid.

In addition, many other facts are known about the parameter $\mathrm{mr}_+(G)$. For example, it is easy to verify that for any graph G, $\mathrm{mr}(G) \leq \mathrm{mr}_+(G) \leq \mathrm{cc}(G)$, where $\mathrm{cc}(G)$ denotes the clique cover number of G (that is, the fewest number of cliques needed to cover the edges of G). Furthermore, if G is known to be chordal (no induced cycles of length four or more), then $\mathrm{mr}_+(G) = \mathrm{cc}(G)$. (See [22] for a proof over the complex numbers. This equation over the reals then follows easily.) However, $\mathrm{mr}(G) < \mathrm{cc}(G)$ for any chordal graph for which it is known that $\mathrm{mr}(G) < \mathrm{mr}_+(G)$, such as a tree that is not a path.

In addition, many other interesting facts are known about the minimum semidefinite rank, including:

- If G' is obtained from G by an edge subdivision, then $\mathrm{mr}_+(G') = \mathrm{mr}_+(G) + 1$ (see [52], a similar argument applies over $\mathbb{R}$),
- If G is triangle free, then $\mathrm{mr}_+(G) \geq \mathrm{mr}_+^{\mathbb{C}}(G) \geq \lceil n/2 \rceil$ (see [28, 29]),
- If G is outerplanar, then $\mathrm{M}_+(G)$ is equal to the tree cover number of G (see [12]).

3.1 Delta conjecture

As mentioned in Section 2, at the AIM workshop in 2006 an interesting inequality was conjectured to hold between the minimum degree and maximum nullity (see [23]). Since that time the validity of this inequality is still unresolved. However, there is significant positive evidence to suggest that the inequality is indeed valid. The *delta conjecture*, as it has become known, states that any graph G with minimum degree $\delta(G)$ satisfies,

$$\mathrm{M}(G) \geq \delta(G).$$

Equivalently, we could ask if $\mathrm{mr}(G) \leq |G| - \delta(G)$ holds for all graphs G.

At present, the delta conjecture is known to hold for many classes of graphs including trees, graphs with $\delta(G) \leq 3$, bipartite graphs (see [21]), along with various other examples.

A stronger version of the delta conjecture involving positive semidefinite matrices has also been suggested and at present remains open (see also [57] for a reference to a conjecture made by Maehara). Is it true that for all graphs G, $\mathrm{M}_+(G) \geq \delta(G)$? If this inequality holds, then the delta conjecture would be solved, as $\mathrm{M}(G) \geq \mathrm{M}_+(G)$. However, at present the relationship between $\mathrm{M}_+(G)$ and $\delta(G)$ has not been fleshed out, and still remains for the most part a mystery. For example, it is not known if $\mathrm{M}_+(G) \geq \delta(G)$ for bipartite graphs G.

On the other hand, there is a nice connection between $\mathrm{M}_+(G)$ and the vertex connectivity of a graph, denoted by $\kappa(G)$. In [57], it was shown that $\mathrm{M}_+(G) \geq \kappa(G)$. Unfortunately, it is also known that $\delta(G) \geq \kappa(G)$ and strict inequality is possible. Recall that the Colin de Verdière parameter $\nu(G)$ (see [36] for a basic introduction on this topic) is defined to be the maximum multiplicity of 0 as an eigenvalue among matrices $A \in S_n(\mathbb{R})$ that satisfy:

- $\mathcal{G}(A) = G$.
- A is positive semidefinite.
- A satisfies the Strong Arnold Hypothesis.

In [47] is was observed that results in [57] in fact implied that $\nu(G) \geq \kappa(G)$. In [8] it is conjectured that $\nu(G) \geq \delta(G)$.

3.2 Graph Complement Conjecture (GCC)

Another interesting conjecture that arose from the 2006 AIM workshop has become known as the graph complement conjecture or GCC for short (see [23]). The GCC can be written as the following conjecture about the minimum rank of G and its complement,

$$\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) \leq |G| + 2, \tag{2}$$

where $\overline{G}$ is the complement of G.

For instance, if $G = C_5$, the cycle on 5 vertices, then $\mathrm{mr}(C_5) = 3$ and $\mathrm{mr}(\overline{C_5}) = \mathrm{mr}(C_5) = 3$. Hence, $\mathrm{mr}(G) + \mathrm{mr}(\overline{G}) = 3 + 3 < 5 + 2$. For paths on n vertices, it can be shown that equality holds in (2) whenever $n \geq 4$ (see [2]).

As with the delta conjecture, there is overwhelming evidence in favor of GCC, however it remains unresolved at present. In addition, stronger forms of GCC have since been suspected and remain open. For example, is the inequality

$$\mathrm{mr}_+(G) + \mathrm{mr}_+(\overline{G}) \leq |G| + 2$$

valid in general?

Observe that GCC (and it variants) can also be stated equivalently in terms of maximum nullities. For example,

$$\mathrm{M}(G) + \mathrm{M}(\overline{G}) \geq |G| - 2, \text{ and } \mathrm{M}_+(G) + \mathrm{M}_+(\overline{G}) \geq |G| - 2.$$

A further strengthening has also been conjectured in terms of the Colin de Verdière parameter $\nu(G)$ (see [9]):

$$\nu(G) + \nu(\overline{G}) \geq |G| - 2.$$

In the recent work [9] there is a number of positive results pertaining to the GCC and it variants, including the case of the join of two graphs and restrictions to k-trees.

4 Zero forcing parameters

One approach to studying the minimum rank or maximum nullity of a graph is to investigate the possible structure of the null space in order to provide bounds on the nullity itself.

For example, if the null space of a given $n \times n$ matrix A has dimension at least 2 (or > 1), then for each $i = 1, 2, \ldots, n$, there exists a nonzero vector $\mathbf{x}$ in the null space of A with $x_i = 0$. Another way to view this concept is the following: Suppose there exists an index i such that any null vector $\mathbf{x}$ with $x_i = 0$ implies $\mathbf{x} = 0$. Then we may conclude that the dimension of null A cannot be more than one. More generally, if a set S of indices has the property that $A\mathbf{x} = 0$ and $x_i = 0$ for all $i \in S$ implies $\mathbf{x} = 0$, then null $A \leq |S|$.

Consider the path on n vertices as a preliminary example. Suppose $A \in \mathcal{S}(P_n)$, and that the vertices of P_n are labeled in increasing order. Suppose that $\mathbf{x}$ is a null vector for A and that $x_1 = 0$. Then the equation $A\mathbf{x} = 0$ in the first coordinate becomes

$$a_{11}x_1 + \sum_{1 \sim j} a_{1j}x_j = a_{11}x_1 + a_{12}x_2 = 0,$$

where $i \sim j$ means vertex i is adjacent to vertex j. The above equations imply that $x_2 = 0$ as $a_{12} \neq 0$. Replacing $i = 1$ with $i = 2$ and continuing in the same manner we deduce that $x_3 = 0$. In other words, if $A \in \mathcal{S}(P_n)$, then the dimension of null A is at most 1. Hence we may conclude that $\mathrm{M}(P_n) = 1$.

More generally, if $A \in \mathcal{S}(G)$, then for each i the ith coordinate of the equation $A\mathbf{x} = 0$ may be written as

$$a_{ii}x_i + \sum_{i \sim j} a_{ij}x_j = 0. \tag{3}$$

Appeal to (3) to provide some intuition as to when a collection of zero coordinates in a null vector of A necessarily implies that the null vector must have been the zero vector to start with. For instance, suppose $x_i = 0$ and $x_j = 0$ for all but one neighbor of i. Then by (3), we have that all of the neighbors of i will have zero coordinates in $\mathbf{x}$. If this process could continue to demonstrate that $\mathbf{x} = 0$, then we may conclude that the dimension of null A cannot exceed the number of neighbors of i. To formalize this idea, we devise a coloring scheme on the vertices of G.

Suppose $G = (V, E)$ is a given graph and that the vertices of G are partitioned into two sets, $V = B \cup W$, where the vertices in B are colored black and the vertices in W are colored white. The goal of the game is to color all of the vertices in G black. To do this, we define a rule known as a *color change rule*. The color change rule in this case, denoted by CCR-Z, is as follows: a black vertex v can color a white neighbor u if it is the only such white neighbor of v. In this case, we say that *v forces u*. The rule corresponds to the implication that we observed above in (3), if we associate the black vertices in B with the initial zero coordinates of a given null vector.

Furthermore, a subset of vertices $S \subset V$ is called a *zero forcing set for G* if whenever the vertices of S are colored black while all remaining all colored white, then all vertices of V are forced to be black under repeated application of the color change rule CCR-Z. For example, a pendent vertex of a path is a zero forcing set for that path. If G is the Petersen graph shown in Figure 2, then the vertices colored black form a zero forcing set.

In other words a zero forcing set of vertices corresponds to an initial collection of indices with the property that if the coordinates of these indices are assigned with zeros in a null vector, then the associated null vector must be the zero vector.

A subset of the vertices is called a *minimum zero forcing set for G* if it is a zero forcing set for G and there is no other zero forcing sets that consist of fewer vertices. For example, a pendant vertex of a path is a minimum zero forcing set of a path, and the set of five black vertices in the Petersen graph above form a minimum zero forcing set for the Petersen graph. Finally, the size of

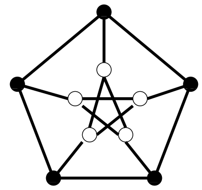

Figure 2: Petersen graph

a minimum zero forcing set for G is called the *zero forcing number of* G, and is denoted by $\mathrm{Z}(G)$ [2]. Thus $\mathrm{Z}(P_n) = 1$ and the zero forcing number of the Petersen graph is 5.

From the construction of $\mathrm{Z}(G)$ it follows that the zero forcing number of a graph is always an upper bound for the maximum nullity of G.

Theorem 4.1. [2] *For any graph G, $\mathrm{Z}(G) \geq \mathrm{M}(G)$.*

Other properties of $\mathrm{Z}(G)$ can be found in [7] including non-uniqueness of minimum zero forcing sets, the intersection over all minimum zero forcing sets of a graph is always empty, and $\mathrm{Z}(G)$ is always an upper bound for $\mathrm{P}(G)$ (the path cover number of G). Other properties of $\mathrm{Z}(G)$ can be found in the works [35, 63, 65].

The case of equality between $\mathrm{Z}(G)$ and $\mathrm{M}(G)$ is still very much of interest and unresolved, namely the problem of characterizing the graphs G for which $\mathrm{M}(G) = \mathrm{Z}(G)$. For example, equality holds between these two parameters for trees and various other examples [2, 49]. However, it is known that the gap between $\mathrm{Z}(G)$ and $\mathrm{M}(G)$ can grow without bound on a sequence of graphs.

The idea of zero forcing on a graph was introduced independently by physicists to study control of quantum systems [24, 66]. Vertices are colored black or white and the same color change rule is applied but the process is called *propagation*, and again it is the minimum number of vertices in a set that propagates (i.e., the zero forcing number $\mathrm{Z}(G)$) that is of interest. Quoting from [24], "Our goal is to determine if such a configuration [of black vertices] is compatible with being a nontrivial eigenstate of the network Hamiltonian (that is an eigenstate in which not all the vertices are black); if not, then the whole network can be controlled." in this context, "nontrivial" refers to non-constant spin orientation. There seem to be deep connections between minimum rank problems and control of quantum systems that are only beginning to be explored.

To bound the maximum nullity of different sets of matrices described by a graph (such as positive definite matrices), variations of zero forcing have been defined by varying the color change rule as needed. Given a color change rule CCR-x and a coloring of of a graph G, the *derived set* is the set of black vertices obtained by applying CCR-x until no more changes are possible. A *(CCR-x) zero forcing set* for G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, then the derived set is all the vertices of G. The *(CCR-x) zero forcing number* is the minimum of $|Z|$ over all (CCR-x) zero forcing sets $Z \subseteq V(G)$.

4.1 Positive semidefinite zero forcing

The analogous concept of zero forcing in the positive semidefinite case comes with its own version of a color change rule. The *positive semidefinite color change rule* [7] is:

CCR-Z_+ Let B be the set consisting of all the black vertices of G. Let $W_1, \ldots, W_k$ be the sets of vertices of the k components of $G - B$ (note that it is possible that $k = 1$). Let $w \in W_i$. If

$u \in B$ and w is the only white neighbor of u in $G[W_i \cup B]$, then change the color of w to black.

As indicated above, the *positive semidefinite zero forcing number of a graph G*, denoted by $\mathrm{Z}_+(G)$, is the minimum of $|X|$ over all CCR-Z_+ zero forcing sets $X \subseteq V_G$.

Forcing with the positive semidefinite color change rule can be viewed as decomposing the graph into a union of certain induced subgraphs and then using CCR-Z on each of these induced subgraphs. For example, it is evident that $\mathrm{Z}_+(T) = 1$ for any tree T, because any one vertex is a positive semidefinite zero forcing set for T. In addition, it is also easy to verify that $\mathrm{Z}_+(G) \leq \mathrm{Z}(G)$ for any graph G.

The graph G in Figure 3 satisfies $\mathrm{Z}_+(G) = 3 < 4 = \mathrm{Z}(G)$ [7]; the vertices colored black form a minimum positive semidefinite zero forcing set.

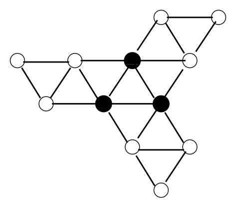

Figure 3: The Pinwheel on 12 vertices

As with the case of standard zero forcing, the positive semidefinite zero forcing number is always an upper bound on the positive semidefinite maximum nullity.

Theorem 4.2. [7] *For any graph G, $\mathrm{Z}_+(G) \geq \mathrm{M}_+(G)$.*

We also note that the concept of positive semidefinite zero forcing is related to the notion of ordered sets that appear in [39, 52, 62]. In fact, it is known (see [7]) that for any graph $G = (V, E)$ and any ordered set S, $V \setminus S$ is a positive semidefinite forcing set for G, and for any positive semidefinite forcing set X for G, there is an order that makes $V \setminus X$ an ordered set for G. Thus $\mathrm{Z}_+(G) + OS(G) = |G|$ (here $OS(G)$ is the ordered set number of G, see [39]). It is also known that $\mathrm{Z}(G)$ is related in a similar manner to the connected ordered set number [62].

From the relation $\mathrm{Z}_+(G) + OS(G) = |G|$ and the fact that $OS(G) \leq |G| - \delta(G)$ from [62], for any graph G we have

$$\mathrm{Z}_+(G) \geq \delta(G),$$

4.2 Other zero forcing parameters

In an attempt to obtain improved bounds on M, graphs that allow loops have been considered [8]. A *loop graph* is a graph that allows loops, i.e., $\widehat{G} = (V_{\widehat{G}}, E_{\widehat{G}})$ where $V_{\widehat{G}}$ is the set of vertices of $\widehat{G}$ and the set of edges $E_{\widehat{G}}$ is a set of two-element multisets. Vertex u is a *neighbor* of vertex v in $\widehat{G}$ if $\{u, v\} \in E_{\widehat{G}}$; note that u is a neighbor of itself if and only if the loop $\{u, u\}$ is an edge. The *underlying simple graph* of a loop graph $\widehat{G}$ is the graph G obtained from $\widehat{G}$ by deleting all loops.

The *set of symmetric matrices described by a loop graph $\widehat{G}$* is

$$\mathcal{S}(\widehat{G}) = \{A = [a_{ij}] \in S_n(\mathbb{R}) : a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E_{\widehat{G}}\}.$$

Note that a loop graph $\widehat{G}$ constrains the zero-nonzero pattern of the main diagonal entries of matrices described by $\widehat{G}$. There is a distinction between a graph, i.e., a simple graph, and a loop graph that has no loops—the latter forces the matrices to have zero diagonal, whereas the former does not (see also Section 5.2). The color change rule for loop graphs is:

CCR-Z($\widehat{G}$) If exactly one neighbor w of u is white, then change the color of w to black.

The *zero forcing number of a loop graph* $\widehat{G}$, denoted by $\mathrm{Z}(\widehat{G})$, is the zero forcing parameter for CCR-Z($\widehat{G}$). The *enhanced zero forcing number* of a (simple) graph G, denoted by $\widehat{\mathrm{Z}}(G)$, is the maximum of $\mathrm{Z}(\widehat{G})$ over all loop graphs $\widehat{G}$ such that the underlying simple graph of $\widehat{G}$ is G (see [8]).

Theorem 4.3. [8] *For any graph G,* $\mathrm{M}(G) \leq \widehat{\mathrm{Z}}(G) \leq \mathrm{Z}(G)$.

Finally, the *loop zero forcing number* of a (simple) graph G, denoted by $\mathrm{Z}_\ell(G)$, is $\mathrm{Z}(\widehat{G})$ where $\widehat{G}$ is the specific loop graph whose underlying simple graph is G, and such that $\widehat{G}$ has a loop at $v \in V_G$ if and only if $\deg_G v \geq 1$.

Although Z_ℓ is already defined as Z evaluated on a specific loop graph, we can see that Z_ℓ is a zero forcing parameter, which aids in computing the value of this parameter. The color change rule associated with the loop zero forcing number is (see also [8]):

CCR-Z_ℓ If u is black and exactly one neighbor w of u is white, then change the color of w to black. If w is white, w has a neighbor, and every neighbor of w is black, then change the color of w to black.

Theorem 4.4. [8] *For any graph G,* $\mathrm{Z}_+(G) \leq \mathrm{Z}_\ell(G) \leq \widehat{\mathrm{Z}}(G)$.

Figure 4: Relationships between zero forcing parameters, parameters related to maximum nullity, and other graph parameters.

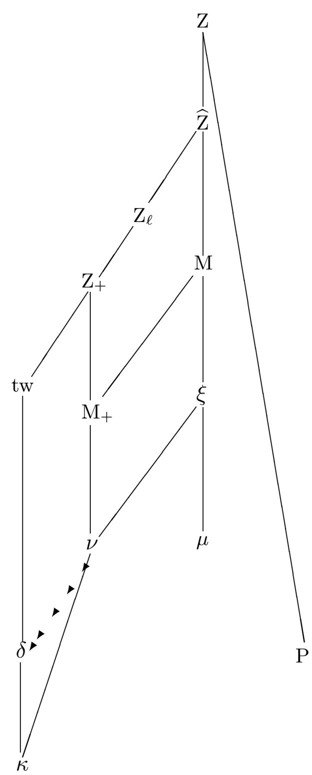

Figure 4, adapted from [8], describes the relationships between the zero forcing parameters, maximum nullity parameters, and other graph parameters (for graphs that have at least one edge). In Figure 4, a line between two parameters q, p means that for all graphs G, $q(G) \leq p(G)$, where q is below p in the diagram. Furthermore, it is known in all cases that inequalities represented in Figure 4 can be strict (see [8]). The strongest form of the delta conjecture ($\delta(G) \leq \nu(G)$) appears as a dashed line of small triangles. (The parameters μ and ξ are Colin de Verdière type parameters and are defined in [36]; $\mathrm{tw}(G)$ denotes the tree-width of G (see [8]).)

5 Minimum rank of patterns and other types of graphs

The families of matrices discussed in previous sections have had off-diagonal nonzero patterns described by edges of simple undirected graphs. In this section we survey work on the minimum rank of matrices with more general patterns of nonzero entries, sometimes eliminating the requirement of positional symmetry by using directed graphs, allowing the pattern to (more fully) constrain the diagonal, and including sign patterns in addition to nonzero patterns. The minimum rank problem for nonzero patterns has been studied over fields other than the real numbers, but for simplicity we limit the discussion here to matrices over the real numbers.

A *nonzero pattern* is an $m \times n$ matrix Y whose entries are elements of $\{*, 0\}$. For $B = [b_{ij}] \in \mathbb{R}^{m\times n}$, the *pattern of* B, $\mathcal{Y}(B) = [y_{ij}]$, is the $m \times n$ nonzero pattern with $y_{ij} = *$ if $b_{ij} \neq 0$ and $y_{ij} = 0$ if $b_{ij} = 0$. A *sign pattern* is a matrix having entries in $\{+, -, 0\}$. For $B \in \mathbb{R}^{m\times n}$, $\mathrm{sgn}(B)$ is the sign pattern having entries that are the signs of the corresponding entries in B. An $n \times n$ (nonzero or sign) pattern is called *square*.

The definitions of *minimum rank* and *maximum nullity* are also extended to an $m \times n$ nonzero pattern or sign pattern. For a nonzero pattern Y:

$$\begin{aligned} \mathrm{mr}(Y) &= \min\{\mathrm{rank}(B) : B \in \mathbb{R}^{m\times n},\ \mathcal{Y}(B) = Y\}. \\ \mathrm{M}(Y) &= \max\{\mathrm{null}(B) : B \in \mathbb{R}^{m\times n},\ \mathcal{Y}(B) = Y\}. \end{aligned}$$

For a sign pattern Y, replace $\mathcal{Y}(B) = Y$ by $\mathrm{sgn}(Y) = Y$. If Y is $m \times n$, then $\mathrm{mr}(Y) + \mathrm{M}(Y) = n$. The problem of determining the minimum rank of a sign pattern, also called the *sign-rank*, has important applications to communication complexity (see Section 5.3).

5.1 Parameters related to minimum rank of nonzero patterns

In [11] it is shown that the minimum rank problem for a nonzero pattern can be converted to a (larger) minimum rank problem of standard type, i.e., symmetric matrices described by a simple undirected graph.

A *t-triangle* of an $m\times n$ nonzero pattern Y is a $t\times t$ subpattern that is permutation similar to a pattern that is upper triangular with all diagonal entries nonzero. The *triangle number* of pattern Y, denoted $\mathrm{tri}(Y)$, is the maximum size of a triangle in Y. The triangle number and t-triangles have been used as a lower bound for minimum rank in both the symmetric and asymmetric minimum rank problems, see e.g., [17], [26]. The triangle number was a focus of the papers [26, 54], where it was denoted $\mathrm{MT}(Y)$. Small patterns Y for which $\mathrm{mr}(Y) = \mathrm{tri}(Y)$ were determined; this includes all $m \times n$ patterns with $m \leq 5$ (the smallest known example where $\mathrm{mr}(Y) > \mathrm{tri}(Y)$ is 7×7).

For a square nonzero pattern Y, the *(row) edit distance to nonsingularity*, $\mathrm{ED}(Y)$, of Y is the minimum number of rows that must be changed to obtain a pattern that requires nonsingularity [11]. The edit distance to nonsingularity is related to the triangle number.

Theorem 5.1. [11] *For an $n \times n$ nonzero pattern Y,* $\mathrm{tri}(Y) + \mathrm{ED}(Y) = n$.

5.2 Graphs of various types

Graphs continue to be a powerful tool in the study of minimum rank of nonzero patterns, but the expansion of the type of pattern discussed necessitates being more inclusive in our definition of "graph." Throughout the remainder of Section 5, a graph can be simple or allow loops, and can be undirected or directed. When describing a specific type of graph, we always use one of the terms *simple* or *loop* and one of the terms *graph* or *digraph*. We use the term *graph of any type* to mean one of a simple graph, a loop graph, a simple digraph, or a loop digraph. We continue to require symmetric matrices for an (undirected) graph (simple or having loops), so in case this restriction is not desired, a doubly directed digraph (simple or having loops) should be used if the pattern of nonzero entries is symmetric. Note that loop graphs were already introduced in Section 4.2 (where a loop graph was denoted by $\widehat{G}$), and the definitions given in that section for the set of matrices described by the graph, minimum rank, maximum nullity, zero forcing number, etc. coincide with those given here, although the notation is slightly different.

Each type of graph describes a set of matrices, the *qualitative class* of G of order n, denoted by $\mathcal{Q}(G)$.

- For a simple graph G, $\mathcal{Q}(G) = \{A \in \mathbb{R}^{n\times n} : A^T = A \text{ and for } i \neq j,\ a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E(G)\}$.
- For a simple digraph G, $\mathcal{Q}(G) = \{A \in \mathbb{R}^{n\times n} : \text{for } i \neq j,\ a_{ij} \neq 0 \Leftrightarrow (i,j) \in E(G)\}$.
- For a loop graph G, $\mathcal{Q}(G) = \{A \in \mathbb{R}^{n\times n} : A^T = A \text{ and } a_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E(G)\}$.
- For a loop digraph G, $\mathcal{Q}(G) = \{A \in \mathbb{R}^{n\times n} : a_{ij} \neq 0 \Leftrightarrow (i,j) \in E(G)\}$.

For a graph G of any type,

$$\mathrm{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{Q}(G)\} \qquad \text{and} \qquad \mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{Q}(G)\}.$$

Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$.

The definition of zero forcing number has been be extended from simple graphs to loop graphs, loop digraphs, and simple digraphs [11, 43]. In this section, we denote a graph by G even if it is a loop graph (or digraph), and the zero forcing number of G is denoted $\mathrm{Z}(G)$. As noted in Section 4.2, the only change needed in the definition of zero forcing number is the color change rule, which depends on the type of graph. The color change rules for a simple graph and a loop graph are CCR-Z and CCR-Z($\widehat{G}$), respectively, defined in Section 4. For simple and loop digraphs, the color change rules are:

CCR-Z(Γ) Let G be a a simple digraph. If u is a black vertex and exactly one out-neighbor v of u is white, then change the color of v to black.

CCR-Z($\widehat{\Gamma}$) Let G be a a loop digraph. If exactly one out-neighbor v of u is white, then change the color of v to black (the possibility that $u = v$ is permitted).

Examples of zero forcing on various types of graphs are given in [43]. Regardless of the type of graph, the zero forcing number bounds maximum nullity from above.

Theorem 5.2. [43] *If G is any type of graph, then* $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

If G is a loop digraph, the nonzero pattern of G is $\mathcal{Y}(G) = \mathcal{Y}(B)$ where $B \in \mathcal{Q}(G)$, the triangle number of G is $\mathrm{tri}(G) = \mathrm{tri}(\mathcal{Y}(G))$, and the edit distance of G is $\mathrm{ED}(G) = \mathrm{ED}(\mathcal{Y}(G))$. These parameters are related.

Theorem 5.3. [11] *If G is a loop digraph, then* $\mathrm{tri}(G) + \mathrm{Z}(G) = |G|$ *and* $\mathrm{ED}(G) = \mathrm{Z}(G)$.

5.3 Minimum rank of sign patterns

The minimum rank of full sign patterns has important applications to communication complexity in computer science (a sign pattern is *full* if all entries are nonzero), and significant progress on minimum rank of full sign patterns has been obtained through work on communication complexity.

In a simple model of communication, described in [27], there are two processors A and B, each of which receives its own input (a string of bits that are 0 or 1), and the goal is to compute a value that is a function of both inputs. The computation function can be described by a $\{0,1\}$-matrix M with rows indexed by the possible inputs of A, columns indexed by the possible inputs for B, and the entry representing the value computed. A *(deterministic) protocol* tells the processors how to exchange information to enable this computation, The *(deterministic) communication complexity* $c(M)$ associated to the $\{0,1\}$ function matrix M is the minimum number of bits that must be transmitted in any protocol associated with M. Melhorn and Schmidt [59] showed that $\log_2 \operatorname{rank} M \le c(M) \le \operatorname{rank} M$ [27].

Communication complexity is also studied from a probabilistic point of view; this approach is described in [60]. An *unbounded error probabilistic protocol* tells the processors how to exchange information to enable computation that will be accurate with probability $> \frac{1}{2}$. The *unbounded error probabilistic communication complexity* upp-cc(M) associated to the function matrix M is the minimum number of bits that must be transmitted in any unbounded error probabilistic protocol associated with M. When studying upp-cc, it is common to use a $\{+1,-1\}$-matrix. A $\{0,1\}$-matrix M can be converted to a $\{+1,-1\}$-matrix by replacing entry m_{ij} by $(-1)^{m_{ij}}$, or equivalently, using $J-2M$, where J is the all ones matrix. If M is an $m\times n$ $\{+1,-1\}$-matrix, then $\operatorname{sgn}(M)$ is a full sign pattern, and if X is an $m\times n$ $\{+,-\}$ sign pattern, then M_X denotes the $m\times n$ $\{+1,-1\}$-matrix having $\operatorname{sgn}(M_X)=X$. For an $\{+1,-1\}$-matrix M, the *sign rank* of M is sign-rank$(M) = \operatorname{mr}(\operatorname{sgn}(M))$. Paturi and Simon [64], [60, p. 106] showed that

$$\log_2 \text{sign-rank}(M) \le \text{upp-cc}(M) \le \log_2 \text{sign-rank}(M) + 1.$$

Thus the computation of sign-rank$(M) = \operatorname{mr}(\operatorname{sgn}(M))$ is of interest in the study of communication complexity. A more thorough introduction to communication complexity and sign-rank its connections to minimum rank are provided by Srinivasan's survey [68] and Lokam's book [60].

Forster [38] established an important lower bound on the sign-rank of an $m\times n$ $\{+1,-1\}$-matrix.

Theorem 5.4. [38] *If M is an $m\times n$ $\{+1,-1\}$-matrix, then*

$$\text{sign-rank}(M) \ge \frac{\sqrt{mn}}{\|M\|},$$

where $\|M\|$ is the spectral norm of M.

An $n\times n$ Hadamard matrix H realizes sign-rank$(H) \ge \frac{n}{\sqrt{n}} = \sqrt{n}$ [38].

Some of the techniques described in Sections 5.1 and 5.2 for nonzero patterns and loop digraphs (which are equivalent to square nonzero patterns) have been adapted to sign patterns. Triangle number used literally is less useful than the following generalization. An $n\times n$ sign pattern X is *sign nonsingular* (SNS) if every $n\times n$ real matrix B such that $\operatorname{sgn}(B)=X$ is nonsingular. The *SNS number* of a sign pattern X, denoted $\operatorname{SNS}(X)$, is the maximum size of an SNS sign pattern submatrix of X [44]. For a square sign pattern X, the *(row) edit distance to nonsingularity*, $\operatorname{ED}(X)$, of X is the minimum number of rows that must be changed to obtain an SNS pattern [44].

Theorem 5.5. [44] *For any $n\times n$ sign pattern X, $\operatorname{SNS}(X)+\operatorname{ED}(X)=n$.*

Sign patterns for which the minimum rank differs from the maximum rank by a fixed amount (such as 1) are discussed in [6].

5.4 Trees

Trees were the first family of simple graphs for which the minimum rank problem was studied, and the minimum rank problem has been solved for square nonzero patterns and square sign patterns for which the graph (simple or loop, undirected or directed) of the nonzero positions is a tree. Minimum rank/maximum nullity can be computed by computing other parameters that are equal for trees. Since solving the minimum rank problem on connected components solves the problem, "tree" can be replaced with "forest" throughout this discussion.

A *simple tree* is a connected acyclic simple graph. A *pseudocycle* is a digraph from which a cycle of length at least three can be obtained by reversing the direction of zero or more arcs. A *ditree* is a (simple or loop) digraph that does not contain any pseudocycles. A *tree* is a graph that is one of the following: a simple tree; a loop graph that is a simple tree after all loops are removed; a ditree. The loop digraph $\mathcal{G}(X)$ of an $n \times n$ sign pattern X is equal to $\mathcal{G}(B)$ for $B \in \mathbb{R}^{n\times n}$ such that $\operatorname{sgn}(B) = X$. A square sign pattern X is a *tree sign pattern* if $\mathcal{G}(X)$ is a ditree.

It is well-known that that $\mathrm{P}(T) = \mathrm{M}(T)$ for a simple tree T. In [11, 43] the definition of path cover number is extended to graphs of other types and the analogous result established for trees of various types. In extending the definition of path cover, there is an issue of whether paths must be induced, which is irrelevant for trees, so here we extend the definition of path cover number only to trees of various types. A loop (di)graph G *requires nonsingularity* if $\mathrm{M}(G) = 0$, i.e., $A \in \mathcal{Q}(G)$ implies A is nonsingular (this is analogous to sign nonsingularity); otherwise G allows singularity. Every simple graph allows singularity, which is immediate by considering $A - \lambda I$ where $A \in \mathcal{Q}(G)$ and λ is an eigenvalue of A. In [11, Definition 4.19], the definition of path cover number was generalized to loop digraphs (and implicitly also to loop graphs) in a manner that retains the property $\mathrm{P}(T) = \mathrm{M}(T)$ for a loop ditree. A key idea was to ignore components that require nonsingularity (such components cannot exist in a simple graph). Let T be a tree of any type. A *path cover* of T is a set of vertex disjoint paths whose deletion from T leaves a graph that requires nonsingularity (or the empty set). The *path cover number* $\mathrm{P}(T)$ is the minimum number of paths in a path cover.

Theorem 5.6. [53, 11, 43, 44] *For a tree of any type or a tree sign pattern,* $\mathrm{M}(T) = \mathrm{P}(T)$.

The parameters $\mathrm{Z}(T)$ and $\mathrm{ED}(T)$ are equal to $\mathrm{M}(T)$ when they have been defined.

Theorem 5.7. [2, 11, 43] *For a tree of any type,* $\mathrm{M}(T) = \mathrm{Z}(T)$.

Theorem 5.8. [11] *For loop ditree,* $\mathrm{M}(T) = \mathrm{ED}(T)$ *and* $\mathrm{mr}(T) = \mathrm{tri}(T)$.

Theorem 5.9. [44] *If T is a tree sign pattern,* $\mathrm{M}(T) = \mathrm{ED}(T)$ *and* $\mathrm{mr}(T) = \mathrm{SNS}(T)$.

For simple trees, the equality $\mathrm{M}(T) = \mathrm{P}(T)$ was established in [53], and was extended to $\mathrm{M}(T) = \mathrm{P}(T) = \mathrm{Z}(T)$ in [2]. The definition of $\mathrm{P}(T)$ was given for loop ditrees in [11], where it was shown that a result in [31] implied $\mathrm{M}(T) = \mathrm{P}(T)$ for loop trees, and $\mathrm{M}(T) = \mathrm{Z}(T) = \mathrm{ED}(T) = \mathrm{P}(T)$ was established for loop ditrees. The equality $\mathrm{M}(T) = \mathrm{Z}(T) = \mathrm{P}(T)$ was extended to simple ditrees in [43] and for sign patterns $\mathrm{M}(T) = \mathrm{ED}(T)$ and $\mathrm{mr}(T) = \mathrm{SNS}(T)$ were established by related methods in [44].

6 Related problems described by (simple) graphs

6.1 Minimum rank over other fields

Recently there has been considerable interest in the study of minimum rank over fields other than the real numbers. For a given graph G of order n, let

$$\mathrm{mr}^F(G) = \min\{\operatorname{rank} A : A \in F^{n\times n}, A^T = A, \mathcal{G}(A) = G\}.$$

Graphs of minimum rank at most 2 over any field F were characterized by a finite set of forbidden induced subgraphs in [17, 18] (with the set of forbidden subgraphs depending on the characteristic of F and number of elements in F). In [34] it was shown that the set of graphs of minimum rank at most k over any finite field is characterized by finitely many forbidden induced subgraphs. In [15] a complete set of forbidden induced subgraphs for minimum rank 3 over $\mathbb{Z}_2$ is determined. In contrast to the finite field case, it is reported that an infinite set of forbidden induced subgraphs is needed to characterize minimum rank 3 over the real numbers [41]. Johnson, Loewy, and Smith characterize graphs having maximum nullity 2 over any infinite field [55].

In 2006 it was an open question whether the minimum rank over another field of characteristic zero (such as $\mathbb{C}$ or $\mathbb{Q}$) could differ from $\mathrm{mr}(G) = \mathrm{mr}^{\mathbb{R}}(G)$ [23]. In [20] examples were given of graphs G_1 and G_2 such that $\mathrm{mr}^{\mathbb{R}}(G_1) > \mathrm{mr}^{\mathbb{C}}(G_1)$ and $\mathrm{mr}^{\mathbb{Q}}(G_2) > \mathrm{mr}^{\mathbb{R}}(G_2)$. Another example of a graph G_3 with $\mathrm{mr}^{\mathbb{Q}}(G_3) > \mathrm{mr}^{\mathbb{R}}(G_3)$ was given in [56]. The graphs G_2 and G_3 provided counterexamples to a conjecture in [5].

A *universally optimal matrix* is a (symmetric) integer matrix A such that every off-diagonal entry of A is 0, 1, or -1 (note for such a matrix $\mathcal{G}(A)$ is independent of field), and for all fields F, $\mathrm{rank}^F(A) = \mathrm{mr}^F(\mathcal{G}(A))$ [30]. In that paper universally optimal matrices were used to show that a number of graphs in the the AIM Minimum Rank Graph Catalog [1] have field independent minimum rank, and examples were presented to show that other graphs in the catalog are field dependent. Additional results on universally optimal matrices and field independence are given in [50].

6.2 The graph parameter $\eta(G)$

If G is a graph on vertices $\{1, \dots, n\}$, the *Haemers number* $\eta(G)$ is defined to be the smallest rank of any $n \times n$ matrix $B = [b_{ij}]$ (over any field) that satisfies $b_{ii} \neq 0$ for $i = 1, \dots, n$ and $b_{ij} = 0$ if i and j are distinct nonadjacent vertices. Clearly $\alpha(G) \leq \eta(G)$ where $\alpha(G)$ is the independence number of G (i.e., the maximum number of vertices with none adjacent). The Laplacian matrix of G shows that $\eta(G) \leq n - c$ where c is the number of connected components of order at least two. Haemers has established a number of properties of $\eta(G)$, including that $\eta(G) \leq \chi(\overline{G})$ (where $\chi(H)$ is the chromatic number of H), and $\eta(G)$ is an upper bound for the Shannon capacity of G [40].

We now examine the relationship between $\eta(G)$ and the minimum rank parameters already discussed. Matrices satisfying the conditions of the Haemers number need not be symmetric but must have positive diagonal entries. If a symmetric matrix $A \in F^{n \times n}$ satisfies the conditions of the Haemers number for G, then $\mathcal{G}(A)$ is a subgraph of G. The Haemers number $\eta(G)$ is not comparable to $\mathrm{mr}(G)$ as the next two examples show.

Example 6.1. It is well known that $\mathrm{mr}(K_{1,3}) = 2$, and $\eta(K_{1,3}) = 3$ because $\alpha(K_{1,3}) = 3$.

Example 6.2. It is well known that $\mathrm{mr}(K_3 \,\square\, P_2) = 3$, where $G \,\square\, H$ denotes the Cartesian product (see [2] for the definition). If we number the vertices so that the two copies of K_3 are numbered $\{1, 2, 3\}$ and $\{4, 5, 6\}$, then we can see that $\eta(K_3 \,\square\, P_2) = 2$ by considering the matrix $J_3 \oplus J_3$ (where J_3 is the 3×3 matrix having every entry equal to 1).

If G is a connected graph, then any matrix $A \in \mathcal{H}_+(G)$ (see Section 3) satisfies the conditions on the matrices used to determine $\eta(G)$, so for a connected graph G, $\eta(G) \leq \mathrm{mr}_+^{\mathbb{C}}(G)$. A somewhat better upper bound for $\eta(G)$ is given by the asymmetric minimum rank of a loop digraph (see Section 5.2) obtained from G by replacing each edge by both arcs and adding a loop at each vertex, but this bound still requires a nonzero entry where an edge is present in the graph, and the Haemers number does not. Recall that the (edge) clique cover number $\mathrm{cc}(G)$ provides an upper bound for $\mathrm{mr}_+(G)$. The *vertex clique cover number*, i.e., the minimum number of cliques needed to cover all the vertices in G, is clearly an upper bound for $\eta(G)$; this was used in Example

6.2. The vertex clique cover number can be much smaller than minimum rank. For example, $\mathrm{mr}(K_n \,\square\, P_2) = n$ (and this does not change if asymmetric matrices are allowed), but the vertices of $K_n \,\square\, P_2$ can be covered by 2 cliques.

6.3 Inverse inertia problem

Barioli and Fallat [10] introduced the term *inertia balanced* to describe a graph with the property that there is a matrix that realizes the minimum rank of the graph and has the number of negative eigenvalues equal to or one less than the number of positive eigenvalues. Inertia balanced graphs played a crucial role in their study of the minimum rank of joins, and they showed that many graphs are inertia balanced. They asked whether all graphs are inertia balanced. Barrett, Hall, and Loewy [16] answered this question in the negative by exhibiting an example of a graph that is not inertia balanced.

In [16] they also began the study of the *inverse inertia problem*, i.e the question of determining what inertias are possible for matrices described by the graph. For a given graph G, inverse inertia problem for G lies in between the minimum rank problem for G and the inverse eigenvalue problem for G, i.e., the question of what spectra are possible for a matrix described by G. Barrett, Hall, and Loewy [16] solved the inverse inertia problem for trees and provide a cut-vertex reduction formula for inverse inertia. The inverse inertia problem is solved for graphs of order at most 6 in [19], where additional techniques for determining inverse inertias are also presented.

6.4 Minimum skew rank

The majority of the work on minimum rank and related problems has focused on symmetric matrices. There has also been work on matrices having a nonzero pattern described by a digraph, or having signs described by a sign pattern, see Section 5. Recently there has also been interest in the problem of ranks of skew-symmetric matrices described by a graph. Such ranks are necessarily even, but full rank may not be possible (even in the case where the order of the graph is even). Let $\mathrm{mr}_-(G)$ (respectively, $\mathrm{MR}_-(G)$) denote the minimum rank (maximum rank) of matrices in the family $\mathcal{S}_-(G)$ of real skew-symmetric matrices whose off-diagonal pattern of nonzero entries is described by the edges of G. A *matching* is a set of edges such that all the vertices are distinct, $\mathrm{match}(G)$ denotes the number of edges in a maximum matching of G, and a matching is *perfect* if it includes every vertex.

Theorem 6.3. [51] *Let G be a graph.*

1. *Every even rank between* $\mathrm{mr}_-(G)$ *and* $\mathrm{MR}_-(G)$ *can be realized.*
2. $\mathrm{mr}_-(G) = |G|$ *if and only if G has a unique perfect matching.*
3. $\mathrm{MR}_-(G) = 2\,\mathrm{match}(G)$*.*
4. *If T is a tree, then* $\mathrm{mr}_-(T) = 2\,\mathrm{match}(T) = \mathrm{MR}_-(T)$*.*
5. $\mathrm{mr}_-(G) = 2$ *if and only if G is a complete multipartite graph.*

Minimum skew rank is computed for several families of graphs, the skew zero forcing number is defined, and related results over fields other than the real numbers are also presented in [51].

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The list above is intended to be used in conjunction with the bibliography in [36], and references cited there are include here only if they are cited in this paper.