Introduction to Combinatorial Matrix Theory

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Matrix Notation

All matrices discussed are real.

Submatrices

- The principal submatrix $A[\alpha]$ consists of the entries in rows and columns in α (i.e., delete rows and columns not in α).
- The principal submatrix obtained by deleting row and column k is denoted $A(k)$.

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} \quad A(2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A[\{1,2\}] = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

Combinatorial Matrix Theory

- Studies patterns of entries in a matrix rather than values.
- In some applications, only the sign of the entry (or whether it is nonzero) is known, not the numerical value.
- In other cases, some entries are missing.
- Uses graphs or digraphs to describe patterns.

Applications: Sign Patterns

- Study of sign patterns arose in economics and has applications to ecology.
- Behavior of supply and demand equilibrium or predator-prey systems under perturbation can be predicted by a sign pattern matrix of partial derivatives whose signs are known.
- Sign stable patterns guarantee stability.
- Sign nonsingular (SNS) patterns guarantee invertibility and unique solution of a system.
- SNS patterns were first studied in the Dimer Problem in statistical mechanics involving the bonding of atoms in a molecule.

Applications: Matrix Completions

- Useful when some data unavailable but full matrix has certain type.
- Used in seismology and geophysics (positive definite matrices).
- Biomolecular modeling (Euclidean distance matrices).
- Image enhancement.

Eigenvalue Interlacing

Theorem

If A is a real symmetric matrix, then the eigenvalues of $A(k)$ interlace the eigenvalues of A.

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{bmatrix} \quad A(2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$\sigma(A) = \{5, 3.56155, -0.561553, -2\}$ $\quad$ $\sigma(A(2)) = \{4, 3, -2\}$

$$5 \geq 4 \geq 3.56155 \geq 3 \geq -0.561553 \geq -2 \geq -2$$

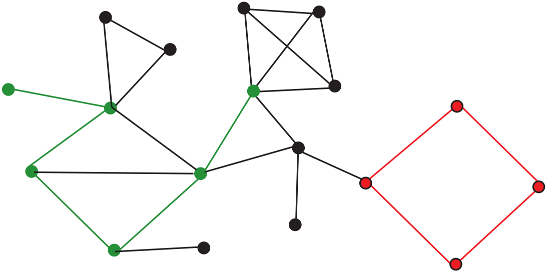
Graph Terminology

We will associate matrices with graphs and digraphs in various ways.

For the minimum rank problem we use a graph to describe the zero-nonzero pattern of entries in a symmetric matrix.

- A (simple) graph $G = (V, E)$ has a (finite, nonempty) set V of vertices and a set E of edges (two element subsets of vertices).
- A graph of does not allow loops or multiple edges.

- A path is a sequence of distinct vertices $v_0, v_1, v_2, \ldots, v_k$ such that for $i = 1, \ldots, k$, $v_{i-1}v_i$ is an edge.
- A cycle is a sequence of vertices $v_1, v_2, \ldots, v_k, v_1$ such that $i \neq j \Rightarrow v_i \neq v_j$ and $v_k v_1$ is an edge, $v_{i-1}v_i$ is an edge $\forall i = 2, \ldots, k$.

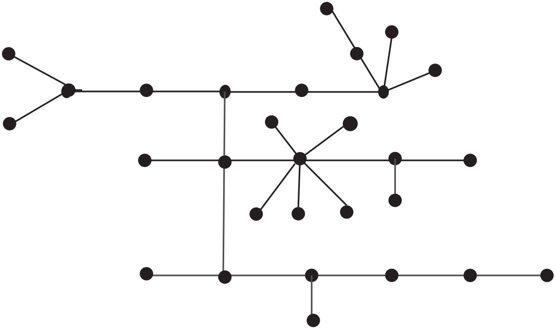


Example:

A path and a cycle.

- A graph is connected if there is a path from any vertex to any other vertex.
- A graph is acyclic if it has no cycles.
- A graph T is a tree if T is connected and T is acyclic.

Example:



A tree.

Matrices and graphs

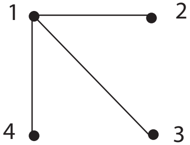
Let A be a symmetric $n \times n$ matrix.

- The graph $\mathcal{G}(A)$ of A is the graph having
  - vertices $1, \ldots, n$
  - edge $\{i, j\}$ of $\mathcal{G}(A)$ if and only if $i \neq j$ and $a_{ij} \neq 0$.
- The diagonal is ignored.

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{G}(A)$



Minimum Rank Problem

Let G a graph.

- $\mathcal{S}(G) = \{A : A \text{ is a symmetric matrix and } \mathcal{G}(A) = G\}$.
- The minimum rank of G is

$$\mathrm{mr}(G) = \min_{A \in \mathcal{S}(G)} \{\mathrm{rank}(A)\}.$$

For a path, $\mathrm{mr}(P_n) = n - 1$.

Example: $*$ is nonzero, $\#$ is indefinite

$$A = \begin{bmatrix} \# & * & 0 & \dots & 0 & 0 \\ * & \# & * & \dots & 0 & 0 \\ 0 & * & \# & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \# & * \\ 0 & 0 & 0 & \dots & * & \# \end{bmatrix}$$

Let each diagonal entry be the negative of the sum of off-diagonal entries in that row.

For a complete graph, $\mathrm{mr}(K_n) = 1$.

Example: $*$ is nonzero, $\#$ is indefinite

$$A = \begin{bmatrix} \# & * & * & \dots & * & * \\ * & \# & * & \dots & * & * \\ * & * & \# & \dots & * & * \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ * & * & * & \dots & \# & * \\ * & * & * & \dots & * & \# \end{bmatrix}$$

Let all entries be 1.

Let A be a symmetric $n \times n$ matrix and G a graph.

- $\text{mult}_\lambda(A)$ denotes the multiplicity of eigenvalue λ of A.
- The maximum multiplicity of an eigenvalue λ in $\mathcal{S}(G)$ is

$$M(G) = \max_{A \in \mathcal{S}(G)} \{\text{mult}_\lambda(A)\}.$$

- The nullity $\mathrm{null}(A)$ is the dimension of the null space of A.
- $M(G) = \max_{A \in \mathcal{S}(G)} \{\mathrm{null}(A)\}$.
- $\text{mr}(G) + M(G) =$ the number of vertices of G.

Minimum Rank Problem for Trees

Let T be a tree. [Nylen LAA 96] gave a method to compute minimum rank of a tree, improved by [Johnson, Leal-Duarte LAMA 99]:

- $\Delta(T) = \max\{p_Q - |Q| : Q \subseteq V(T)$ and $T - Q$ consists of p_Q disjoint paths$\}$.

Theorem

$$|T| - \text{mr}(T) = M(T) = \Delta(T)$$

Algorithm (for computing Δ, and thus M and mr)

Let T be a tree.

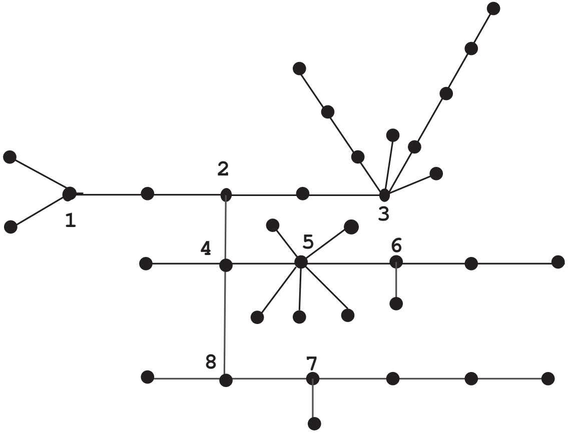
Set $Q = \emptyset$ (set of vertices to be deleted).

Set $H = \{$vertices of T with degree $\geq 3\}$ (candidates for deletion).

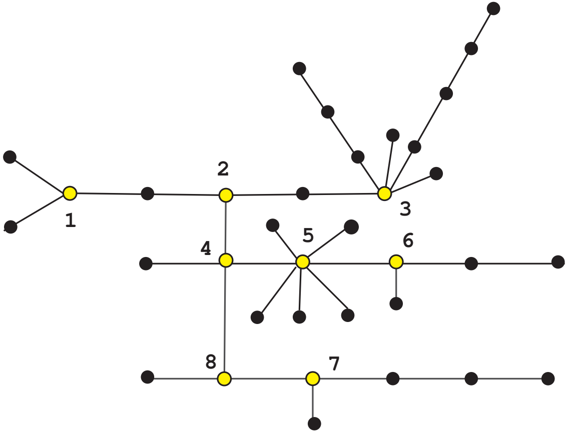
While $H \neq \emptyset$:

1. *Set T_H = the unique component of $T - Q$ that contains an H-vertex.*
2. *Set $W = \{w \in H :$ at most 1 component of $T_H - w$ is not H-free$\}$.*
3. *$Q = Q \cup W$.*
4. *$H = H - W$.*
5. *For each $v \in H$, if $\deg_{T-Q} v \leq 2$, remove v from H.*

- Example: Compute the minimum rank of the tree T by computing $\Delta(T)$ = maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the tree T by computing $\Delta(T) =$ maximum multiplicity of eigenvalue 0:



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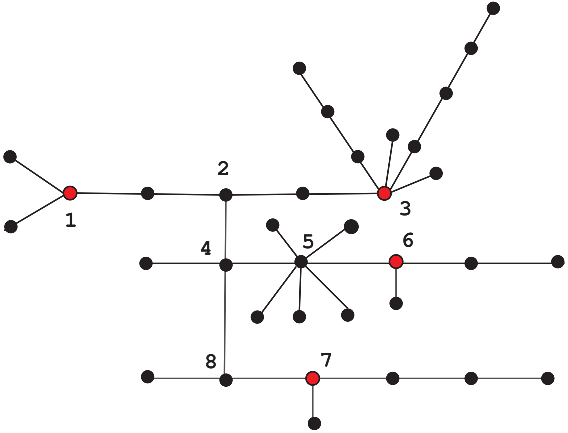

Algorithm (for computing Δ, and thus M and mr)

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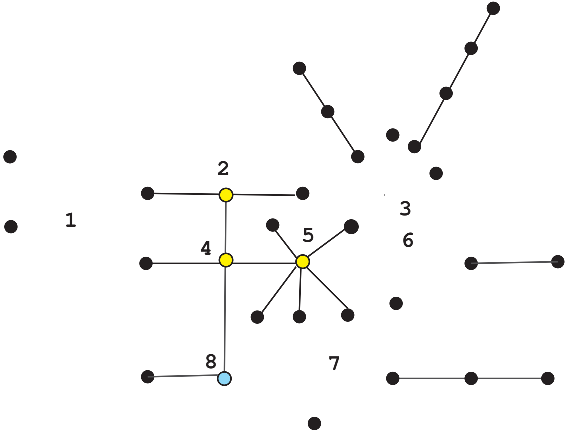
- Example: Compute the minimum rank of the tree T by computing $\Delta(T)$ = maximum multiplicity of eigenvalue 0:



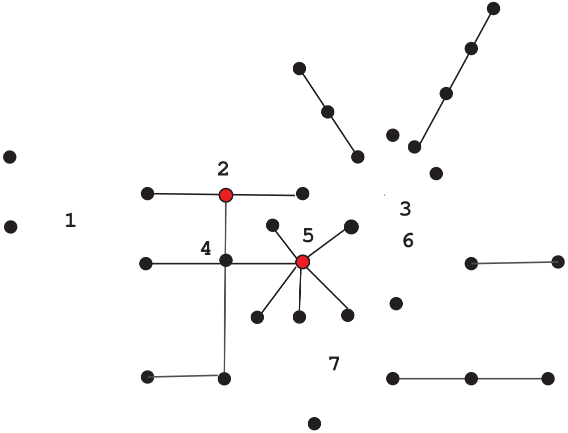
Second Iteration:

Algorithm (for computing Δ, and thus M and mr)

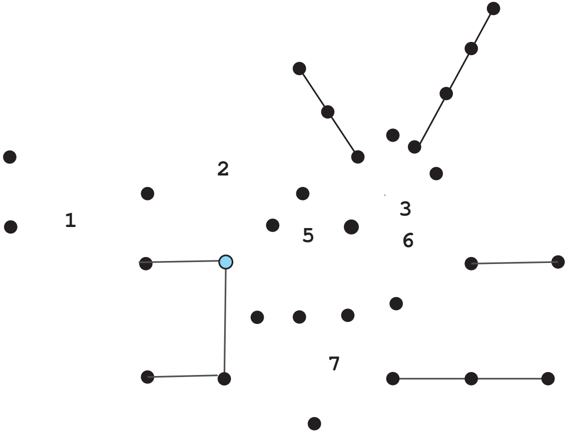
While $H \neq \emptyset$:

1. *Set* T_H = *the unique component of* $T - Q$ *that contains an* H*-vertex.*
2. *Set* $W = \{w \in H :$ *at most 1 component of* $T_H - w$ *is not* H*-free*$\}$.
3. $Q = Q \cup W$.
4. $H = H - W$.
5. *For each* $v \in H$, *if* $\deg_{T-Q} v \leq 2$, *remove* v *from* H.

- Example: Compute the minimum rank of the tree T by computing $\Delta(T)$ = maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the tree T by computing $\Delta(T)$ = maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the tree T by computing maximum multiplicity of eigenvalue 0:

- $Q = \{1, 2, 3, 5, 6, 7\}$, i.e., 6 vertices were deleted.
- There are 18 paths.
- $M(T) = 18 - 6 = 12$.
- $\mathrm{mr}(T) = 35 - 12 = 23$.

Trees are done, but work continues on the minimum rank of graphs, e.g.

- if graph has a cut-vertex, reduce the problem to the pieces [Barioli, Fallat, Hogben 04].
- new Colin de Verdière type parameter used for computation of minimum rank [Barioli, Fallat, Hogben 04], [Hogben, van der Holst 05].
- Many new results were obtained at

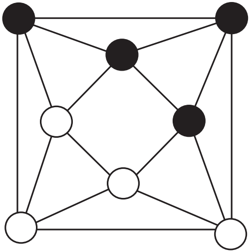
American Institute of Mathematics Workshop
Spectra of families of matrices described by graphs, digraphs, and sign patterns
October 23 to October 27, 2006
http://aimath.org/pastworkshops/matrixspectrum.html

AIM workshop particpants: Francesco Barioli, Wayne Barrett, Avi Berman, Richard Brualdi, Steven Butler, Sebastian Cioaba, Dragoš Cvetković, Jane Day, Louis Deaett, Luz DeAlba, Shaun Fallat, Shmuel Friedland, Chris Godsil, Jason Grout, Willem Haemers, Leslie Hogben, In-Jae Kim, Steve Kirkland, Raphael Loewy, Judith McDonald, Rana Mikkelson, Sivaram Narayan, Olga Pryporova, Uri Rothblum, Irene Sciriha, Bryan Shader, Wasin So, Dragan Stevanović, Pauline van den Driessche, Hein van der Holst, Kevin Vander Meulen, Amy Wangsness, Amy Yielding.

Color change rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example H_8



Color change rule

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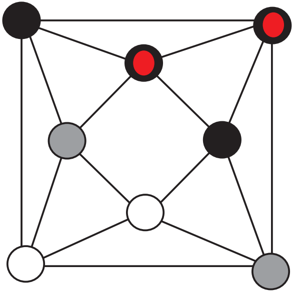
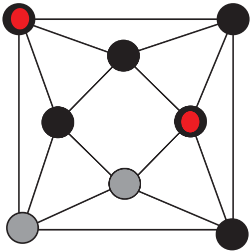
Example (H_8 continued)



Color change rule

If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

Example (H_8 continued)



Zero-Forcing Sets

- Given a coloring of G, the derived coloring is the result of applying the color-change rule until no more changes result.
- A zero forcing set for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.
- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

Theorem

[AIM LAA 07] *For any graph G, $M(G) \leq Z(G)$.*

Example $M(H_8) \leq Z(H_8) \leq 4$ and so $\text{mr}(H_8) \geq 4$.
In fact $\text{mr}(H_8) = 4$.

More information about minimum rank problems and a bibliography can be found in

Fallat and Hogben, The Minimum Rank of Symmetric Matrices Described by a Graph: A Survey, *Linear Algebra and Its Applications*, 426 (2007) 558-582.

Matrix Completion Problems

- A partial matrix is a square array in which some entries are specified and others are not.
- A completion of a partial matrix is a choice of values for the unspecified entries.

Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

B is a partial matrix and A is a completion of B.

All matrices discussed are real and square.

All classes of matrices discussed are generalizations of the positive definite matrices.

Classes of matrices to be discussed:

The following are equivalent (positive definite):

- A is symmetric and for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$.
- A is symmetric and all eigenvalues are positive.
- A is symmetric and all principal minors are positive.

Classes of matrices to be discussed:

A is a P-matrix if all principal minors are positive.

Example:

$$A = \begin{bmatrix} 5 & 1 & -1 \\ 0 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix} \text{ is a } P \text{ matrix.}$$

$$\det A = 14, \quad \det A[\{1,2\}] = \det \begin{bmatrix} 5 & 1 \\ 0 & 1 \end{bmatrix} = 5, \text{ etc.}$$

Classes of matrices to be discussed:

A matrix is a Z-matrix if all off-diagonal entries are nonpositive.

The following are equivalent for real matrices (A is an M-matrix):

- A is a Z-matrix and all eigenvalues have positive real part.
- A is a Z-matrix and all principal minors are positive.
- $A = sI - P$ where $P \geq 0$ and $s > \rho(P)$.
- A is a Z-matrix and $A^{-1} \geq 0$.

Classes of matrices to be discussed:

A matrix A is an inverse M-matrix if A^{-1} is an M-matrix.

Example:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -2 & 4 & -1 \\ 0 & -2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{5}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

A is an M-matrix and A^{-1} is an inverse M-matrix.

- All classes X of matrices discussed are hereditary, i.e. if A is an X-matrix then every principal submatrix of A is an X-matrix.

- If X is hereditary, in order for a partial matrix B to have an X-completion, it is necessary that every fully specified principal submatrix of B is an X-matrix and any necessary sign conditions are satisfied.

- A partial matrix B is a partial X-matrix if every fully specified principal submatrix of B is an X-matrix and any necessary sign conditions are satisfied.

- a pattern of specified entries has the X-completion property if every partial X-matrix B described by that pattern can be completed to an X-matrix.

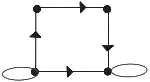
Digraph Terminology

For matrix completions we use a digraph to describe the pattern of specified entries in a matrix.

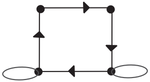
- A digraph $D = (V, E)$ has a (finite, nonempty) set V of vertices and a set E of arcs (ordered pairs of vertices).
- A digraph of does not allow multiple edges but is allowed to have both arcs (v, w) and (w, v) and does allow loops (arcs of the form (v, v)).
- A digraph $D = (V, E)$ is symmetric if $(v, w) \in E$ implies $(w, v) \in E$.
- The subdigraph induced by $W \subseteq V$ is the digraph $D[W] = (W, E \cap (W \times W))$.

- A path is a sequence of distinct vertices $v_0, v_1, v_2, \ldots, v_k$ such that for $i = 1, \ldots, k$, (v_{i-1}, v_i) is an arc.
- A cycle is a sequence of vertices $v_1, v_2, \ldots, v_k, v_1$ such that $i \neq j \Rightarrow v_i \neq v_j$ and (v_k, v_1) is an arc, (v_{i-1}, v_i) is an arc $\forall i = 2, \ldots, k$.
- A digraph is strongly connected if there is a path from any vertex to any other vertex.
- A digraph is connected if it is all one piece (there is an un-oriented path from any vertex to any other vertex).

Example:



connected
but not strongly connected



strongly connected

- A digraph is nonseparable if there is no vertex whose deletion disconnects the digraph.
- A block of a digraph is a maximal nonseparable subdigraph.
- A digraph is a clique if every vertex has a loop and for any two distinct vertices u, v, the edge $\{u, v\}$ is present (respectively, both arcs $(u, v), (v, u)$ are present).
- A digraph is block-clique (also called 1-chordal) if every block is a clique.

Partial matrices and digraphs

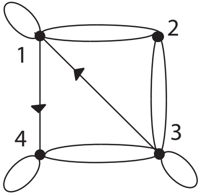
Let B be an $n \times n$ partial matrix.

- The digraph $\mathcal{D}(B)$ of B is the digraph having
  - vertices $1, \ldots, n$
  - arc (i, j) of $\mathcal{D}(B)$ if and only if b_{ij} is specified.

Example:

$$A = \begin{bmatrix} 2 & -1 & ? & 5 \\ -1 & ? & 0 & ? \\ 4 & 8 & -3 & 0 \\ ? & ? & 11 & 2 \end{bmatrix}$$

$\mathcal{D}(B)$

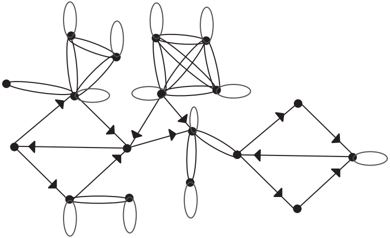


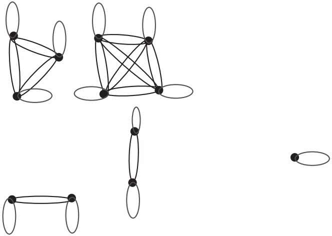
- A digraph D has the X-completion property if every partial X-matrix B such that $\mathcal{D}(B) = D$ can be completed to an X-matrix.

- For all the classes X discussed here, a block diagonal matrix with each diagonal block being an X-matrix is an X matrix.
- Thus the a digraph has the X-matrix property if and only if each connected component (each piece) has the X-matrix property.

- The X-matrix completion problem reduces to the subdigraph induced by the looped vertices if a digraph has the X-matrix property if and only if the subdigraph induced by the looped vertices has the X-matrix property.

Example: If the X-completion problem reduces to the subdigraph induced by the looped vertices, the left digraph





has the X-completion property.

- A class X has the triangular property if whenever A is a block triangular matrix and every diagonal block is an X-matrix, then A is an X matrix.

- If X has the triangular property, B is a partial matrix in block triangular form (as a pattern), and each diagonal block can be completed to an X-matrix, then B can be completed to an X-matrix.

- If X has the triangular property and is closed under permutation similarity, then a digraph D has the X-completion property if and only if every strongly connected component of D has the X-completion property, and we say the problem reduces to strongly connected components.

The Positive Definite Matrix Completion Problem

[Grone, Johnson, Sa, Wolkewicz LAA 84]

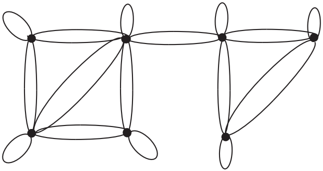
- Only symmetric digraphs are considered.
- The positive definite matrix completion problem reduces to the subdigraph induced by the looped vertices.

Theorem

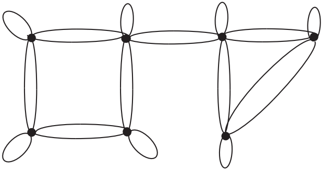
A symmetric digraph having a loop at every vertex has the positive definite completion property if and only if it is chordal (any cycle of length ≥ 4 has a chord).

A "symmetric digraph having all loops" is also described as a "(simple) graph" and all diagonal entries assumed specified.

Example: This digraph has the positive definite completion property.



Example: This digraph does not have the positive definite completion property.



The M-matrix Completion Problem

[Hogben 98]

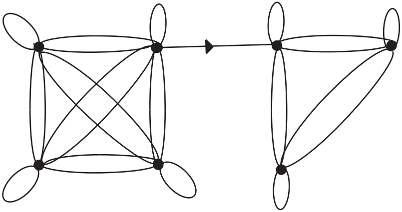
- The M-matrix completion problem reduces to the subdigraph induced by the looped vertices.
- The M-matrix completion problem reduces to the strongly connected components.

Theorem

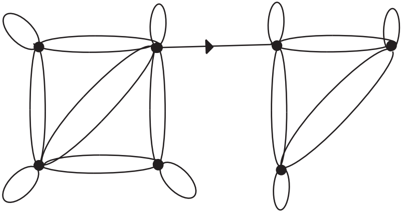
A digraph having a loop at every vertex has the M-matrix completion property if and only every strongly connected component is a clique (equivalently, a partial matrix is permutation similar to a triangular matrix with complete diagonal blocks).

This result is of lesser importance since any partial M-matrix can be tested for M-completion by using the 0-completion.

Example: This digraph has the M-matrix completion property.



Example: This digraph does not have the M-matrix completion property.



The Inverse M-matrix Completion Problem

- The inverse M-matrix completion problem does NOT reduce to the subdigraph induced by the looped vertices.
- A digraph is cycle-clique if the induced subdigraph of every cycle is a clique.
- An alternate path to a single arc in a digraph D is a path of length greater than 1 between vertices v and w such that the arc (v, w) is in D.
- A digraph D is path-clique if the induced subdigraph of every alternate path to a single arc is a clique.

Theorem

[Johnson, Smith LAA 96] *A symmetric digraph D with a loop at every vertex has the inverse M-completion property if and only if D is block-clique.*

[Hogben LAA 98] *A digraph D with a loop at every vertex has the inverse M-completion property if and only if D is path-clique and cycle-clique.*

The P-matrix Completion Problem

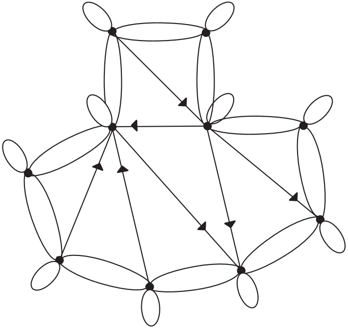
- The P-matrix completion problem reduces to the subdigraph induced by the looped vertices.

Theorem

[Johnson, Kroschel ELA 96] *Every symmetric digraph having a loop at every vertex has the P-matrix completion property*

A family of digraphs that do not have the P-completion property is given in [DeAlba, Hogben LAA 2000].

Example: This digraph does not have the P-matrix completion property.



More information about matrix completions and details of the references can be found at

http://orion.math.iastate.edu/lhogben/MC/homepage.html