Introduction to Combinatorial Matrix Theory

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August 29, 2006

Research partially supported by NSF grant DMS-0353880.

Combinatorial Matrix Theory

- Studies patterns of entries in a matrix rather than values.
- In some applications, only the sign of the entry (or whether it is nonzero) is known, not the numerical value.
- In other cases, some entries are missing.
- Uses graphs or digraphs to describe patterns.

Applications: Sign Patterns

- Study of sign patterns arose in economics and has applications to ecology.
- Behavior of supply and demand equilibrium or predator-prey systems under perturbation can be predicted by a sign pattern matrix of partial derivatives whose signs are known.
- Sign stable patterns guarantee stability.
- Sign nonsingular (SNS) patterns guarantee invertibility and unique solution of a system.
- SNS patterns were first studied in the Dimer Problem in statistical mechanics involving the bonding of atoms in a molecule.

Applications: Matrix Completions

- Useful when some data unavailable but full matrix has certain type.
- Used in seismology and geophysics (positive definite matrices).
- Biomolecular modeling (Euclidean distance matrices).
- Image enhancement.

All (numerical) matrices discussed are real and symmetric, but the ideas can be generalized to nonsymmetric matrices.

Submatrices

- The principal submatrix $A[\alpha]$ consists of the entries in rows and columns in α (i.e., delete rows and columns not in α).
- The principal submatrix obtained by deleting row and column k is denoted $A(k)$.

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{bmatrix} \quad A(2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A[\{1,2\}] = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

Eigenvalue Interlacing

Theorem

If A is a real symmetric matrix, then the eigenvalues of $A(k)$ interlace the eigenvalues of A.

Example:

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 3 & 1 \end{bmatrix} \quad A(2) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

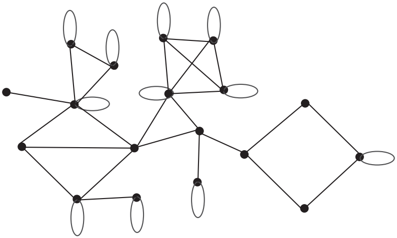
$\sigma(A) = \{5, 3.56155, -0.561553, -2\}$ $\qquad \sigma(A(2)) = \{4, 3, -2\}$

$$5 \geq 4 \geq 3.56155 \geq 3 \geq -0.561553 \geq -2 \geq -2$$

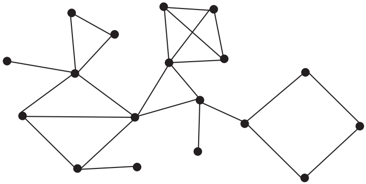
Graph Terminology

- A simple graph $\widehat{G}$ does not allow loops or multiple edges.
- A graph G allows loops but does not allow multiple edges.
- The simple graph associated with G, denoted $\widehat{G}$, is obtained from G by suppressing loops.

Example:



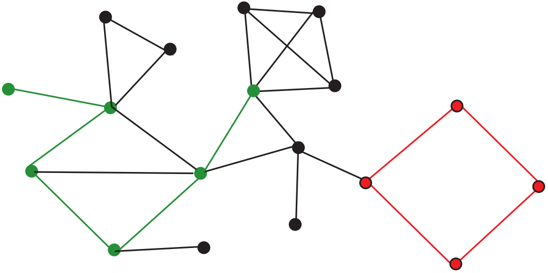
G



$\widehat{G}$

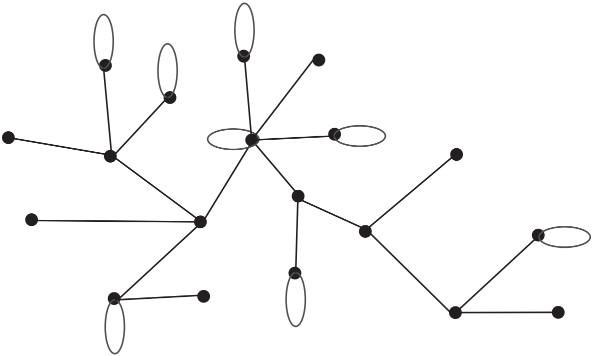
- A path is a sequence of distinct vertices $v_0, v_1, v_2, \ldots, v_k$ such that for $i = 1, \ldots, k$, $v_{i-1}v_i$ is an edge.
- A cycle is a sequence of vertices $v_1, v_2, \ldots, v_k, v_1$ such that $i \neq j \Rightarrow v_i \neq v_j$ and $v_k v_1$ is an edge, $v_{i-1}v_i$ is an edge $\forall i = 2, \ldots, k$.

Example:



A path and a cycle.

- A graph is connected if there is a path from any vertex to any other vertex.
- A graph is acyclic if it has no cycles.
- A graph T is a tree if T is connected and $\widehat{T}$ is acyclic.



Example:

A tree.

Matrices and graphs

- We will associate matrices with graphs in several ways.
- With an $n \times n$ matrix we associate a graph having vertices $\{1, \ldots, n\}$.
- The i, j entry will be associated with the edge (or non-edge) between i and j.

Permutation Similarity and Vertex Numbering

- Applying a permutation similarity to a matrix does not change its eigenvalues or its rank.
- Applying a permutation similarity to a symmetric matrix A corresponds to renumbering the vertices of the graph of A
- Unlabeled graph diagrams can be used.

Block Diagonal Matrices and Graph Components

- If $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, then the spectrum of A is the union of the spectra of $A_1, A_2, \ldots, A_k$ and $\text{rank}(A) = \sum_{i=1}^{k} \text{rank}(A_i)$.
- Any symmetric matrix is permutation similar to a direct sum of irreducible matrices (block diagonal matrix).
- The irreducible summands of A correspond to the connected components of the graph of A.
- Only connected graphs are studied.

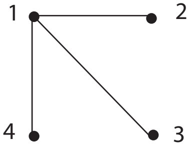
Let A be a symmetric $n \times n$ matrix.

- The simple graph $\widehat{\mathcal{G}}(A)$ of A is the graph having
  - vertices $1, \ldots, n$
  - edge ij of $\widehat{\mathcal{G}}(A)$ if and only if $i \neq j$ and $a_{ij} \neq 0$.
- The diagonal is ignored.

Example:

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

$\widehat{\mathcal{G}}(A)$



Minimum Rank Problems for Simple Graphs

Let A be a symmetric $n \times n$ matrix and $\widehat{G}$ a simple graph.

- $\widehat{\mathcal{S}}(\widehat{G}) = \{A : A \text{ is a symmetric matrix and } \widehat{\mathcal{G}}(A) = \widehat{G}\}$.
- The minimum rank of $\widehat{G}$ is

$$\widehat{\text{mr}}(\widehat{G}) = \min_{A \in \widehat{\mathcal{S}}(\widehat{G})} \{\text{rank}(A)\}.$$

Let A be a symmetric $n \times n$ matrix and $\widehat{G}$ a simple graph.

- $\text{mult}_\lambda(A)$ denotes the multiplicity of eigenvalue λ of A.

- The maximum multiplicity of an eigenvalue λ in $\widehat{\mathcal{S}}(\widehat{G})$ is

$$\widehat{M}(\widehat{G}) = \max_{A \in \widehat{\mathcal{S}}(\widehat{G})} \{\text{mult}_\lambda(A)\}.$$

- $\widehat{\text{mr}}(\widehat{G}) + \widehat{M}(\widehat{G}) =$ the number of vertices of $\widehat{G}$.

Let $\widehat{G}$ have n vertices.

- It is easy to obtain a matrix $A \in \widehat{\mathcal{S}}(\widehat{G})$ with $\operatorname{rank} A = n-1$ (translate).
- It is easy to have all eigenvalues distinct (i.e., $\operatorname{rank} A \geq n-1$).
- For a path, $\widehat{\operatorname{mr}}(\widehat{P}_n) = n-1$.

Example: $\#$ is nonzero, $*$ is indefinite

$$A = \begin{bmatrix} * & \# & 0 & \dots & 0 & 0 \\ \# & * & \# & \dots & 0 & 0 \\ 0 & \# & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & \# \\ 0 & 0 & 0 & \dots & \# & * \end{bmatrix}$$

Minimum Rank Problems for Simple Trees

Let $\widehat{T}$ be a simple tree.

- $\Delta(\widehat{T}) = \max\{p_Q - |Q| : Q \subseteq V(\widehat{T})$ and $\widehat{T} - Q$ consists of p_Q disjoint paths$\}$.

Theorem ([JLD99])

$$|\widehat{T}| - \widehat{\mathrm{mr}}(\widehat{T}) = \widehat{M}(\widehat{T}) = \Delta(\widehat{T})$$

Algorithm (for computing Δ, and thus $\widehat{M}$ and $\widehat{\text{mr}}$)

Let $\widehat{T}$ be a simple tree.

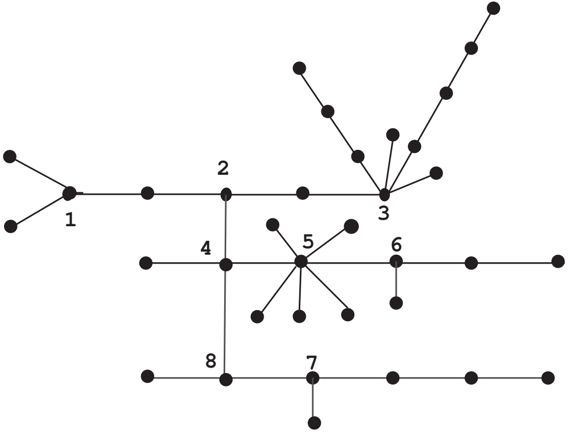
Set $Q = \emptyset$ (set of vertices to be deleted).

Set $H = \{$vertices of $\widehat{T}$ with degree $\geq 3\}$ (candidates for deletion).

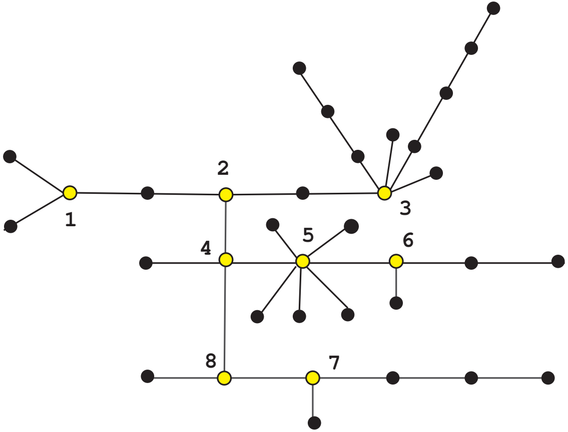
While $H \neq \emptyset$:

1. *Set $\widehat{T}_H =$ the unique component of $\widehat{T} - Q$ that contains an H-vertex.*
2. *Set $W = \{w \in H:$ at most 1 component of $\widehat{T}_H - w$ is not H-free$\}$.*
3. *$Q = Q \cup W$.*
4. *$H = H - W$.*
5. *For each $v \in H$, if $\deg_{\widehat{T}-Q} v \leq 2$, remove v from H.*

- Example: Compute the minimum rank of the tree $\widehat{T}$ by computing $\Delta(\widehat{T})$ = maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the tree $\widehat{T}$ by computing $\Delta(\widehat{T})$ = maximum multiplicity of eigenvalue 0:



Algorithm (for computing Δ, and thus $\widehat{M}$ and $\widehat{\mathrm{mr}}$)

While $H \neq \emptyset$:

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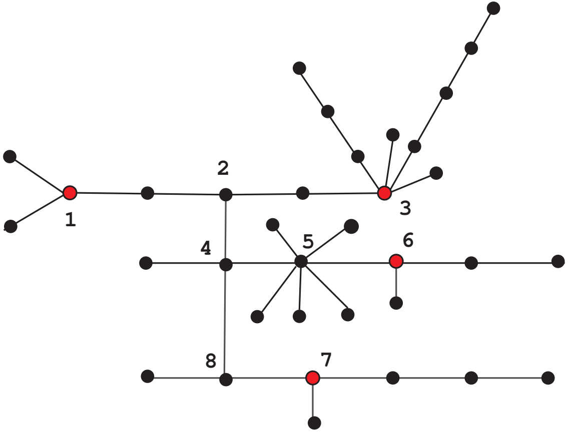

Algorithm (for computing Δ, and thus $\widehat{M}$ and $\widehat{\mathrm{mr}}$)

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3. *$Q = Q \cup W$.*
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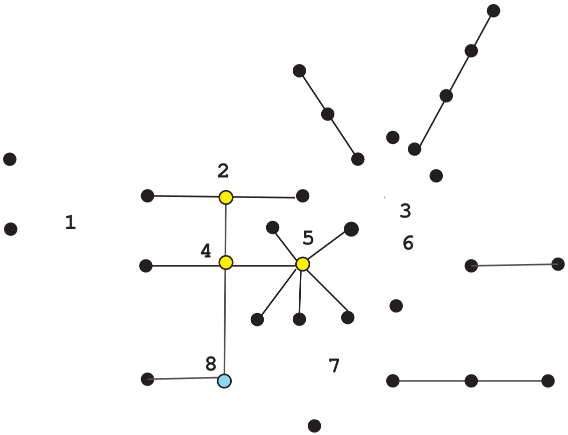

Second Iteration:

Algorithm (for computing Δ, and thus $\widehat{M}$ and $\widehat{\mathrm{mr}}$)

While $H \neq \emptyset$:

1. *Set $\widehat{T}_H$ = the unique component of $\widehat{T} - Q$ that contains an H-vertex.*
2. *Set $W = \{w \in H :$ at most 1 component of $\widehat{T}_H - w$ is not H-free$\}$.*
3. $Q = Q \cup W$.
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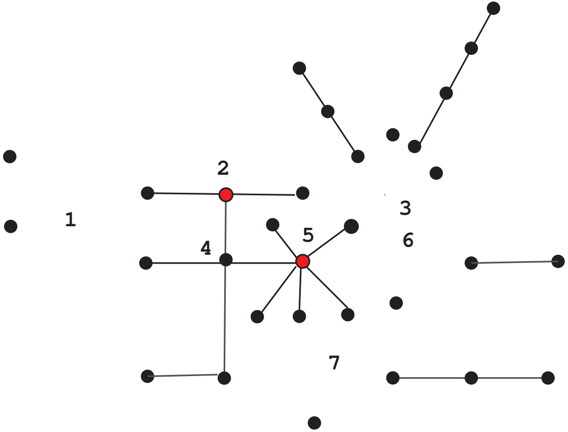
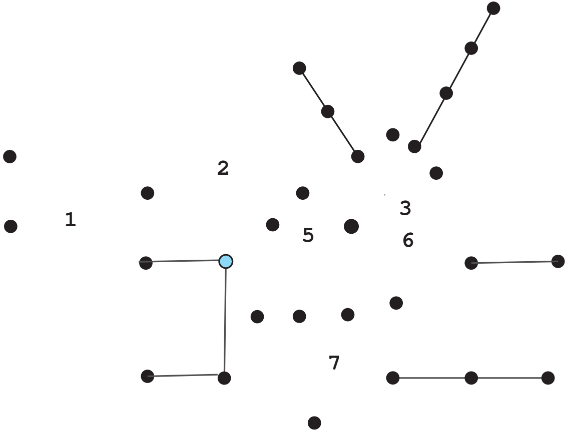


- Example: Compute the minimum rank of the tree $\widehat{T}$ by computing $\Delta(\widehat{T})$ = maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the simple tree $\widehat{T}$ by computing maximum multiplicity of eigenvalue 0:

- $Q = \{1, 2, 3, 5, 6, 7\}$, i.e., 6 vertices were deleted.
- There are 18 paths.
- $\widehat{M}(\widehat{T}) = 18 - 6 = 12$.
- $\widehat{\text{mr}}(\widehat{T}) = 35 - 12 = 23$.

Trees are done, but work continues on

- minimum rank of arbitrary graphs-
  - if graph has a cut-vertex, reduce the problem to the pieces [BFH04].
  - new Colin de Verdière type parameter used for computation of minimum rank [BFH05],[HvdH06]
- matrices over fields other than $\mathbb{R}$
- positive semidefinite minimum rank
- etc.

Sign Patterns

- A sign pattern is an $n \times n$ matrix $Z = [z_{ij}]$ whose entries z_{ij} are elements of $\{+, -, 0\}$.
- The qualitative class or sign pattern class of Z, $\mathcal{Q}(Z) = \{A : sgn(a_{ij}) = z_{ij}, 1 \leq i, j \leq n\}$.
- The graph $\mathcal{G}(Z)$ of the symmetric $n \times n$ sign pattern Z:
  - vertices $1, \ldots, n$
  - ij is an edge of $\mathcal{G}(Z)$ if and only if $z_{ij} \neq 0$.
- The graph has loops.
- A symmetric sign pattern Z is a tree sign pattern (TSP) if $\mathcal{G}(Z)$ is a tree.

Minimum Rank Problems for Tree Sign Patterns

- For symmetric Z, $\mathcal{S}(Z) =$ $\{A : A \text{ is a symmetric matrix and for all } i, j,\ \text{sgn}(a_{ij}) = z_{ij}\}$.
- The diagonal is restricted by the sign pattern.
- The minimum rank of Z is the minimum rank among matrices having sign pattern Z.
- The symmetric minimum rank of symmetric Z is

$$\text{mr}(Z) = \min_{A \in \mathcal{S}(Z)} \{\text{rank}(A)\}.$$

- For a symmetric tree sign pattern, minimum rank equals symmetric minimum rank.

- The maximum multiplicity of an eigenvalue λ in $\mathcal{S}(Z)$ is

$$M_\lambda(Z) = \max_{A \in \mathcal{S}(Z)} \{\text{mult}_\lambda(A)\}.$$

- All positive (respectively, negative) eigenvalues have the same maximum multiplicity.

- $\text{mr}(Z) + M_0(Z) =$ the size of Z.

Let Z be a (symmetric) sign pattern.

- $\mathcal{S}(Z)$ (or $\mathcal{G}(Z)$) allows eigenvalue λ if there is a matrix $A \in \mathcal{S}(Z)$ having eigenvalue λ.
- $\mathcal{S}(Z)$ allows eigenvalue 0 if and only if $\det Z$ is identically 0 or has both positive and negative terms.

Algorithm (Computation of M_λ for TSP [DHHHW06])

Let Z be a TSP and $T = \mathcal{G}(Z)$.

Set $Q = \emptyset$ (set of vertices to be deleted).

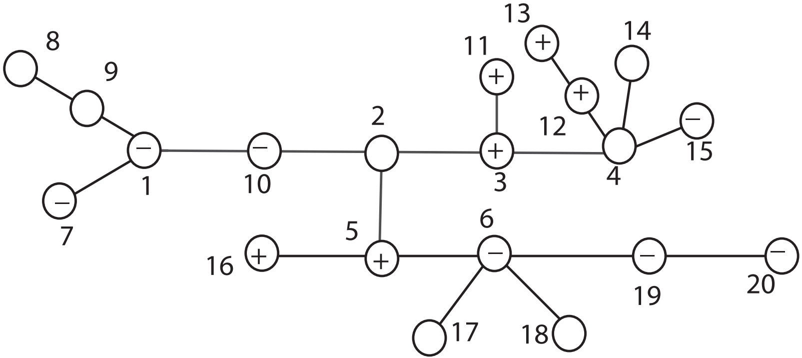
Set $H = \{$vertices of $\widehat{T}$ with degree $\geq 3\}$ (candidates for deletion).

While $H \neq \emptyset$:

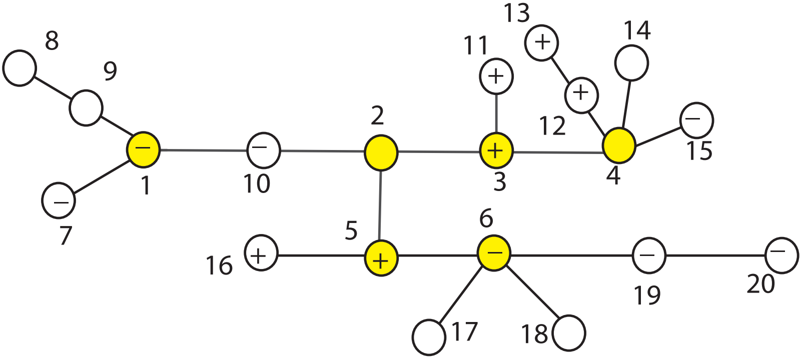
1. *Set $T_H =$ the unique component of $T - Q$ that contains an H-vertex.*
2. *Set $W = \{w \in H :$ at most 1 component of $T_H - w$ is not H-free$\}$.*
3. *For each $w \in W$, if there are at least two H-free components of $T_H - w$ that allow eigenvalue λ, then $Q = Q \cup \{w\}$.*
4. *$H = H - W$.*
5. *For each $v \in H$, if $\deg_{T-Q} v \leq 2$, remove v from H.*

$M_\lambda(Z)$=(number of components of $T - Q$ that allow λ) $- |Q|$.

- Example: Compute the minimum rank of the sign pattern Z by computing maximum multiplicity of eigenvalue 0:



- Example: Compute the minimum rank of the sign pattern Z by computing maximum multiplicity of eigenvalue 0:



Algorithm (Computation of M_0)

While $H \neq \emptyset$:

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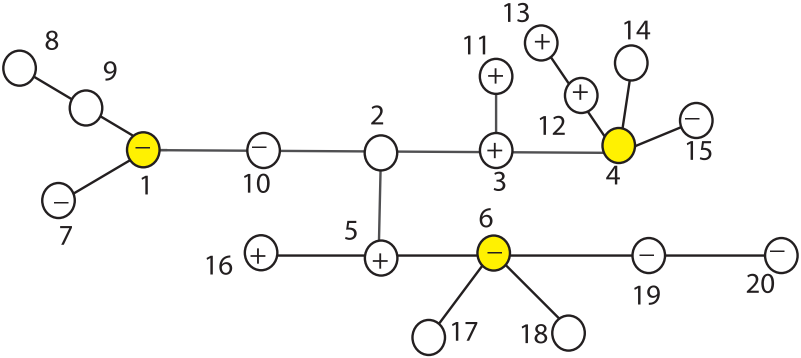

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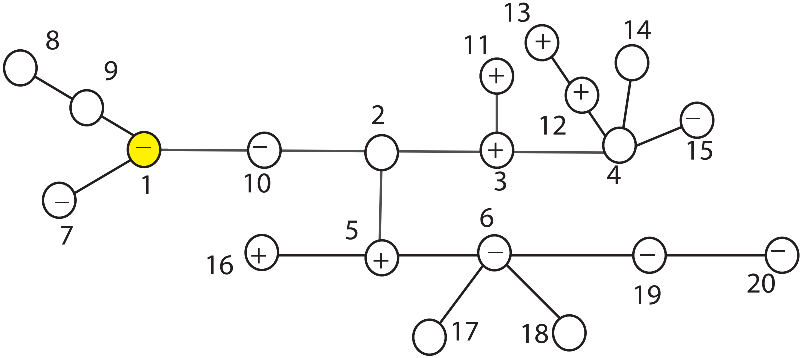

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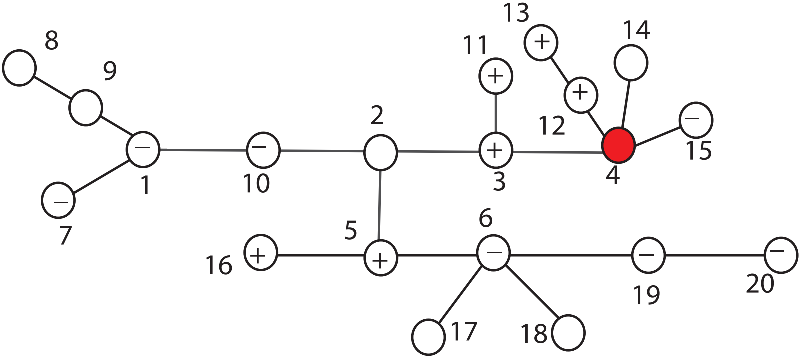

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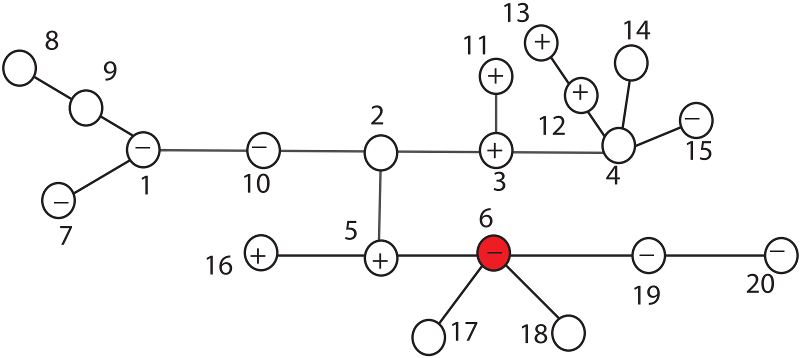

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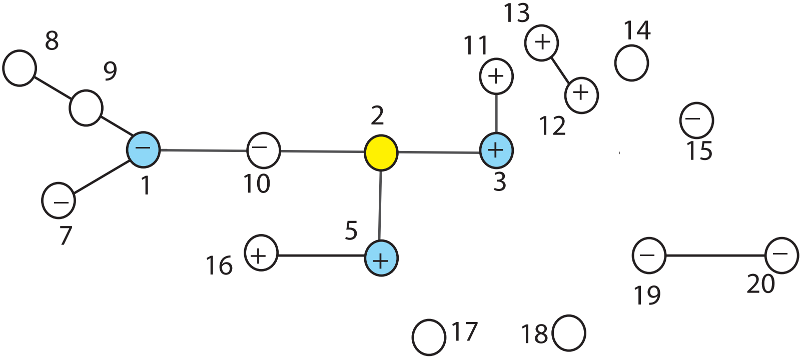

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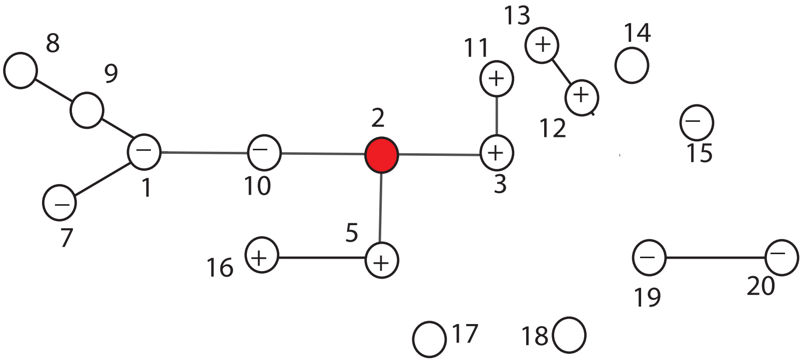

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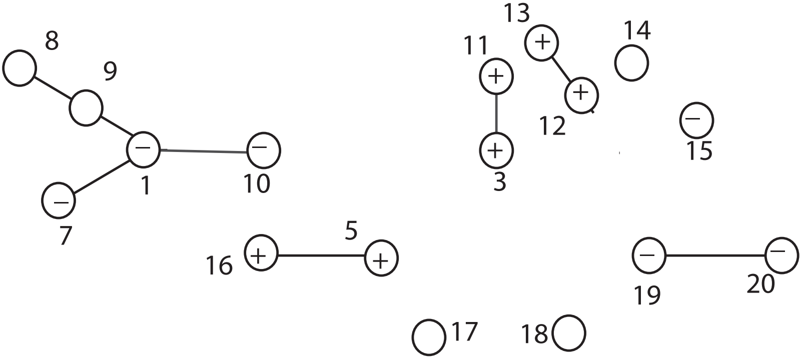


- Example: Compute the minimum rank of the sign pattern Z by computing maximum multiplicity of eigenvalue 0:

- $Q = \{2, 4, 6\}$, i.e., 3 vertices were deleted.
- There are 8 components that allow eigenvalue 0.
- $M_0(Z) = 8 - 3 = 5$.
- $\text{mr}(Z) = 20 - 5 = 15$.

Matrix Completion Problems

- A partial matrix is a square array in which some entries are specified and others are not.
- A completion of a partial matrix is a choice of values for the unspecified entries.

Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

B is a partial matrix and A is a completion of B.

The following are equivalent for real matrices:

- A is symmetric and for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$ (A is positive definite).
- A is symmetric and all eigenvalues are positive.
- A is symmetric and all principal minors are positive.

The following are equivalent for real matrices:

- A is symmetric and for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^T A\mathbf{x} \geq 0$ (A is positive semidefinite).
- A is symmetric and all eigenvalues are nonnegative.
- A is symmetric and all principal minors are nonnegative.

Classes X of matrices to be discussed:

- positive definite matrices
- positive semidefinite matrices
- strictly copositive matrices: A is strictly copositive if A is symmetric and for all $\mathbf{x} \geq 0, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$
- copositive matrices: A is copositive if A is symmetric and for all $\mathbf{x} \geq 0$, $\mathbf{x}^T A\mathbf{x} \geq 0$

Example:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix}$$

is strictly copositive but not positive definite (any positive matrix is strictly copositive).

- A matrix completion problem asks whether a partial matrix, or family of partial matrices with a given pattern of specified entries, has a completion of a specific type X, such as a positive definite matrix.

Example:

$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Matrix A completes B to a positive semidefinite matrix.

- All classes X of matrices discussed are hereditary, i.e. if A is an X-matrix then every principal submatrix of A is an X-matrix.

- If X is hereditary, in order for a partial matrix B to have an X-completion, it is necessary that every fully specified principal submatrix of B is an X-matrix.

- A partial matrix B is a partial X-matrix if every fully specified principal submatrix of B is an X-matrix.

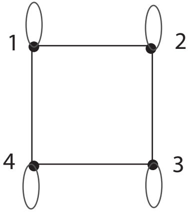
- a pattern of specified entries has the X-completion property if every partial X-matrix B described by that pattern can be completed to an X-matrix.

Graph Theoretic Techniques

- Graphs are used to describe the pattern of specified entries classes of for symmetric matrices such as positive (semi)definite and (strictly) copositive matrices; otherwise digraphs are used.
- The specified entries in partial matrix B are represented by edges in the graph $\mathcal{G}(B)$.

  Example:

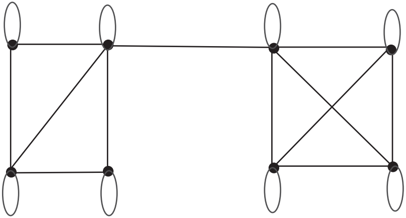
$$B = \begin{bmatrix} 2 & -1 & ? & 0 \\ -1 & 2 & 2 & ? \\ ? & 2 & 3 & 1 \\ 0 & ? & 1 & 1 \end{bmatrix}$$



Theorem ([GJSW84])

- *A graph having a loop at every vertex has the positive definite completion property if and only if it is chordal (any cycle of length ≥ 4 has a chord).*
- *A graph has the positive definite completion property if and only if the subgraph induced by the vertices with loops has the positive definite completion property.*

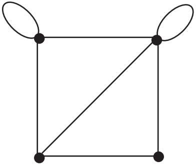
Example:



has the positive definite completion property.

Theorem ([GJSW84])

A graph has the positive semidefinite completion property if and only if each component has all loops or none, and components with loops are chordal.

Example: 

does not have the positive semidefinite completion property.

The (Strictly) Copositive Matrix Completion Problems

- A is strictly copositive if A is symmetric and for all $\mathbf{x} \geq 0, \mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} > 0$.
- A is copositive if A is symmetric and for all $\mathbf{x} \geq 0$, $\mathbf{x}^T A\mathbf{x} \geq 0$.

Example:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

is copositive but not strictly copositive, and not positive semidefinite.

- The partial matrix B is a partial strictly copositive matrix if every fully specified principal submatrix of B is a strictly copositive matrix.

- The partial matrix B is a partial copositive matrix if every fully specified principal submatrix of B is a copositive matrix.

Example:

$$B = \begin{bmatrix} 3 & 1 & x_{13} & -1 \\ 1 & 1 & 2 & x_{24} \\ x_{13} & 2 & 1 & -1 \\ -1 & x_{24} & -1 & 2 \end{bmatrix}$$

is a partial strictly copositive matrix.

Theorem ([HJR05])

Let B be a partial copositive matrix with every diagonal entry specified. For each pair of unspecified off-diagonal entries, set $x_{ij} = x_{ji} = \sqrt{b_{ii}b_{jj}}$. The resulting matrix is copositive, and is strictly copositive if B is a partial strictly copositive matrix.

Example:

$$B = \begin{bmatrix} 3 & 1 & x_{13} & -1 \\ 1 & 1 & 2 & x_{24} \\ x_{13} & 2 & 1 & -1 \\ -1 & x_{24} & -1 & 2 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 1 & \sqrt{3} & -1 \\ 1 & 1 & 2 & \sqrt{2} \\ \sqrt{3} & 2 & 1 & -1 \\ -1 & \sqrt{2} & -1 & 2 \end{bmatrix}$$

A completes B to a strictly copositive matrix.

Theorem ([H06])

Let $B = \begin{bmatrix} x_{11} & \mathbf{b}^T \\ \mathbf{b} & B_1 \end{bmatrix}$ *be a partial strictly copositive* $n \times n$ *matrix having all entries except the 1,1-entry specified. Let* $\|\cdot\|$ *be a vector norm. Complete* B *to a strictly copositive matrix by choosing a value for* x_{11} *as follows:*

1. $\beta = \min_{\mathbf{y} \geq 0, \|\mathbf{y}\|=1} \mathbf{b}^T\mathbf{y}$.
2. $\gamma = \min_{\mathbf{y} \geq 0, \|\mathbf{y}\|=1} \mathbf{y}^T B_1 \mathbf{y}$.
3. $x_{11} > \frac{\beta^2}{\gamma}$.

Corollary

Every partial strictly copositive matrix can be completed to a strictly copositive matrix.

Example: The partial matrix

$$B = \begin{bmatrix} x_{11} & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & x_{55} & x_{56} \\ x_{16} & 1 & -1 & 1 & x_{56} & 3 \end{bmatrix}$$

is a partial strictly copositive matrix.

Select index 5. The only principal submatrix completed by a choice of b_{55} is $B[\{3,5\}]$.
Any value that makes $5x_{55} > (-1)^2$ will work.
Choose $x_{55} = 1$.

Example: The partial matrix

$$B = \begin{bmatrix} x_{11} & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & x_{55} & x_{56} \\ x_{16} & 1 & -1 & 1 & x_{56} & 3 \end{bmatrix}$$

is a partial strictly copositive matrix.
Select index 1. The only principal submatrices completed by a choice of b_{11} are principal submatrices of

$$B[\{1,2,3\}] = \begin{bmatrix} x_{11} & -5 & 1 \\ -5 & 1 & -2 \\ 1 & -2 & 5 \end{bmatrix}.$$

$$B[\{1,2,3\}] = \begin{bmatrix} x_{11} & -5 & 1 \\ -5 & 1 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

Using $\|\cdot\|_1$:

1. $\beta = \min_{||\mathbf{y}||_1=1} \mathbf{b}^T\mathbf{y} = -5$.
2. $\gamma = \min_{||\mathbf{y}||_1=1} \mathbf{y}^T B[\{2,3\}]\mathbf{y} = \frac{1}{10}$.
3. Choose $x_{11} > \frac{\beta^2}{\gamma}$; choose $b_{11} = 256$.

$b_{11} = 256, \; b_{55} = 1.$ $\begin{bmatrix} 256 & -5 & 1 & x_{14} & x_{15} & x_{16} \\ -5 & 1 & -2 & x_{24} & x_{25} & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ x_{14} & x_{24} & 1 & 1 & x_{45} & 1 \\ x_{15} & x_{25} & -1 & x_{45} & 1 & x_{56} \\ x_{16} & 1 & -1 & 1 & x_{56} & 3 \end{bmatrix}$

Set $x_{ij} = x_{ji} = \sqrt{b_{ii} b_{jj}}$.

$$\begin{bmatrix} 256 & -5 & 1 & 16 & 16 & 16\sqrt{3} \\ -5 & 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 5 & 1 & -1 & -1 \\ 16 & 1 & 1 & 1 & 1 & 1 \\ 16 & 1 & -1 & 1 & 1 & \sqrt{3} \\ 16\sqrt{3} & 1 & -1 & 1 & \sqrt{3} & 3 \end{bmatrix}$$

is a strictly copositive matrix.

It is not true that every partial copositive matrix can be completed to a copositive matrix.

Example: $B = \begin{bmatrix} x_{11} & -1 \\ -1 & 0 \end{bmatrix}$ is a partial copositive matrix that cannot be completed to a copositive matrix.

Choose a value for x_{11} (> 0).

If $x_{11} > 0$, then for the vector $\mathbf{x} = [1, x_{11}]^T$, $\mathbf{x}^T B \mathbf{x} = -x_{11}$.

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