Math 510 Examination 1 October 2, 2009

All answers must be justified by computation or explanation. Write each question on a separate page and submit the answers in order. All questions carry equal credit but all parts of a question are not necessarily equal.

1. Let A and B be normal. Show that A and B are similar if and only if they are unitarily similar.

2. Let $A, B \in \mathbb{C}^{n\times n}$.

   (a) Suppose that A is invertible. Show that AB and BA are similar.

   (b) Without using Theorem 2.8 (that AB and BA have the same nonzero eigenvalues) show that for any $A, B \in \mathbb{C}^{n\times n}$, AB and BA have the same characteristic polynomial.

   (c) If both A and B are singular, must AB and BA be similar?

3. Find a unitary matrix U having first column $[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}]^T$.

4. Give a one line proof of, or a counter-example to, each of the following statements, where $A, B \in \mathbb{C}^{n\times n}$:

   (a) If A, B, C, D are all 1×1 matrices, then $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB)$.

   (b) If A, B, C, D are all 2×2 matrices, then $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB)$.

   (c) For the block matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, $\operatorname{rank} \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \geq \operatorname{rank} A + \operatorname{rank} D$.