Math 510 Homework 5 Dec. 8, 2006

Concise, well-written mathematics is valued.

1. Give examples in $\mathbb{C}^{n\times n}$ of each of the following. (Hint: in each case examples exist with $n \leq 4$.)
   (a) Two matrices with the same minimal and characteristic polynomials that are not similar to each other.
   (b) A matrix with an eigenvalue that has geometric multiplicity different from its algebraic multiplicity.
   (c) A nondiagonal, positive definite matrix.
   (d) A normal matrix that is neither unitary, Hermitian nor skew-Hermitian.

2. Let $H = [h_{ij}] \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For each statement, prove or give a counter example.
   (a) All diagonal entries h_{ii} are real.
   (b) For each $i = 1, 2, ..., n$, $\lambda_1 \leq h_{ii} \leq \lambda_n$.
   (c) For all $i, j \in \{1, 2, ..., n\}$, $\lambda_1 \leq \det \begin{bmatrix} h_{ii} & h_{ij} \\ h_{ji} & h_{jj} \end{bmatrix} \leq \lambda_n$.

3. Let N be a normal $n \times n$ complex matrix such that $N^3 - 2I$ is nilpotent. Prove that $N^3 = 2I$.

4. Let $N \in \mathbb{C}^{n\times n}$ be normal. Prove

$$\max_{\mathbf{v}\in\mathbb{C}^n, \|\mathbf{v}\|=1} \mathrm{Re}(\mathbf{v}^* N \mathbf{v}) = \max_{\lambda\in\sigma(N)} \mathrm{Re}(\lambda).$$

5. For what values of $\gamma \in \mathbb{C}$ do there exist nonsingular $A, B \in \mathbb{C}^{n\times n}$ such that $AB = \gamma BA$?