Math 510 Homework 1 Due Wednesday Sept. 8, 2010

Directions: Because some of this homework covers undergraduate linear algebra that has not been carefully developed in this class, it may be unclear what results you can use. Thus I have listed an assortment at the end that I think is sufficient. If you have another result you want to use, ask me by e-mail and I may allow it. Vectors are **bold**.

1. Suppose V is a group with operation $+$ (not assumed abelian), F is a field, and $\cdot$ is an operation that maps $F \times V$ onto V with the following properties for $a, b \in F$ and $\mathbf{v}, \mathbf{u} \in V$:

   - $a \cdot (\mathbf{v} + \mathbf{u}) = a \cdot \mathbf{v} + a \cdot \mathbf{u}$.
   - $(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$.
   - $(ab) \cdot \mathbf{v} = a \cdot (b \cdot \mathbf{v})$.
   - $1 \cdot \mathbf{v} = \mathbf{v}$.

   Show that V is a vector space over F.

2. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, D are square and A is invertible. Prove

$$\det M = \det A \det(D - CA^{-1}B).$$

3. Let V be be a finite dimensional vector space over $\mathbb{R}$ with bilinear form B and let $\mathcal{C} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be an ordered basis. Define ${}_{\mathcal{C}}[B]_{\mathcal{C}}$, the matrix of B with respect to $\mathcal{C}$, to be the $n \times n$ matrix whose i, j-entry is $B(\mathbf{x}_i, \mathbf{x}_j)$. Prove the following theorem (which was stated in lecture):

   Theorem 1.
$$B(\mathbf{v}, \mathbf{w}) = [\mathbf{v}]_{\mathcal{C}}^T \; {}_{\mathcal{C}}[B]_{\mathcal{C}} \; [\mathbf{w}]_{\mathcal{C}}.$$

4. The following theorem was (or will be) stated in lecture:

   Theorem 2. *If V is a finite dimensional inner product space over $\mathbb{R}$ and W is a subspace of V, then*
$$W^{\perp\perp} = W.$$

   Show that this is not true in general without the assumption that V is finite dimensional, i.e., give an example of a V and W such that

$$W \subset W^{\perp\perp}.$$

5. Without using any results about block matrices other than the definition, and using the Laplace expansion definition of the determinant (consistently on the row or column of your choice), prove that if A, D are square matrices over field F and $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ is a block matrix, then

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \det D.$$

Results that may be assumed:

- Any result stated in the lectures unless you are being asked to prove that.
- Any definition from the text.
- Anything in Chapter 1.
- High school algebra and properties of real and complex numbers.
- Basic arithmetic of matrices, including $A\mathbf{0} = \mathbf{0}$ and if $\mathbf{0}$ represents zero matrices of appropriate size, $\mathbf{0}A = \mathbf{0} = A\mathbf{0}$.
- Principle of induction.
- $\det(AB) = \det A \det B$.
- Question 5 may be used in the proof of other questions, but no other question may be used in the proof of 5 as per its directions.