Math 510 Final Examination December 18, 2009

**Directions:** All answers must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. Five complete correct answers will receive full credit, but you may answer one additional question if desired (maximum of 6 will be graded, best 5 scores used). Write each solution on a separate page. Submit solutions in the same order as the questions.

1. Give a list of complex matrices such that every $3 \times 3$ complex matrix $A$ satisfying $A^3 = I$ must be similar to exactly one matrix on your list.

2. Give examples in $\mathbb{C}^{n\times n}$ of each of the following. (Hint: in each case examples exist with $n \leq 4$.)

   (a) Two matrices with the same minimal and characteristic polynomials that are not similar to each other.

   (b) A matrix with an eigenvalue that has geometric multiplicity different from its algebraic multiplicity.

   (c) A nondiagonal, positive definite matrix.

   (d) A normal matrix that is neither unitary, Hermitian nor skew-Hermitian.

3. Let $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and singular values $\sigma_1, \dots, \sigma_n$. Prove

$$\prod_{i=1}^{n} |\lambda_i| = \prod_{i=1}^{n} \sigma_i.$$

4. Let $N$ be a normal $n \times n$ complex matrix such that $N^3 - 2I$ is nilpotent. Prove that $N^3 = 2I$.

5. Let $V$ be the vector space of polynomials over $\mathbb{C}$ of degree $\leq n$. For $0 \leq k \leq n$, define a linear functional $f_k \in V^*$ by $f_k(p) = p(k)$ for $p \in V$. Show that $\{f_0, \dots, f_n\}$ is a basis for $V^*$.

6. Let $V$ be an $n$-dimensional inner product space and let $\mathbf{x}, \mathbf{y}$ be fixed vectors in $V$. Show that $T\mathbf{v} = \langle \mathbf{v}, \mathbf{x} \rangle \, \mathbf{y}$ defines a linear operator $T$ on $V$, and describe its adjoint $T^*$ explicitly.

7. Let $P_1, \dots, P_k \in \mathbb{C}^{n\times n}$ satisfying $\sum_{i=1}^{k} P_i = I$. Prove that the following are equivalent:

   (a) $P_i^2 = P_i, \ i = 1, \dots, k$.

   (b) $P_i P_j = 0, \ i \neq j$.

   (c) $\operatorname{rank} P_1 + \cdots + \operatorname{rank} P_k = n$.

8. Let $A$ and $B$ be $n \times n$ Hermitian matrices, and let $A$ be positive definite. Show that for any $x \in \mathbb{C}^n$,

$$\lambda_{min}(A^{-1}B) \leq \frac{x^* B x}{x^* A x} \leq \lambda_{max}(A^{-1}B).$$

   (You may use the fact that for any real positive definite matrix $M$ there exists a positive definite matrix $S$ such that $M = S^2$.)