

Directions: All answers must be justified by computation or explanation. Greater weight will be given to one whole (correct) solution than to two error-free but incomplete solutions. Six complete correct answers will receive full credit, but you may answer additional questions if desired. Write each solution on a separate page. Submit solutions in the same order as the questions.

1. The characteristic polynomial of $A \in \mathbb{Q}^{11 \times 11}$ is $p(x) = (x-1)^5(x^2+1)^3$, the minimum polynomial of A is $p(x) = (x-1)^3(x^2+1)^2$, and the geometric multiplicity of eigenvalue $\lambda = 1$ is 3. Determine the invariant factors of A .
2. Let A be a normal matrix. Prove that A^* commutes with any matrix that commutes with A .
3. (a) Assume $A \in F^{n \times n}$ is idempotent. Prove that the rank of A is equal to the number of nonzero eigenvalues of A .
(b) Give a counterexample to show that the statement in 3a may be false without the assumption that A is idempotent.
4. Find the reduced singular value decomposition (i.e., find singular values and right and left singular vectors for nonzero singular values) for $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$.
5. Let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian. Prove that $\lambda_{\max}(A - B) + \lambda_{\min}(B) \leq \lambda_{\max}(A)$.
6. Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$. Prove that
$$\text{rank}((A - \lambda I)^2) + \text{rank}((A - \lambda I)^8) \geq \text{rank}((A - \lambda I)^5) + \text{rank}((A - \lambda I)^5).$$
7. Let $A \in F^{n \times n}$ and let $\lambda \in \sigma(A)$. Prove that A is similar to a matrix having the sum of the elements in each row equal to λ .
8. Let $H \in \mathbb{C}^{n \times n}$ be Hermitian and partition H as $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ with H_{11}, H_{22} square. Prove that the Schur complement H/H_{11} is Hermitian.

1. The characteristic polynomial of $A \in \mathbb{Q}^{11 \times 11}$ is $p(x) = (x - 1)^5(x^2 + 1)^3$, the minimum polynomial of A is $p(x) = (x - 1)^3(x^2 + 1)^2$, and the geometric multiplicity of eigenvalue $\lambda = 1$ is 3. Determine the invariant factors version of the rational canonical form of A , i.e., determine $\text{RCF}_{IF}(A)$.

Find Jordan form first:

Eigenvalue 1: Geometric multiplicity is 3, so 3 Jordan blocks of size n_1, n_2, n_3 . Largest block size $n_1 = 3$. Algebraic multiplicity is 5, i.e., $3 + n_2 + n_3 = 5$, i.e., $n_2 = n_3 = 1$. Thus for $\lambda = 1$, $J_3(1) \oplus J_1(1) \oplus J_1(1)$ and the elementary divisors are $(x - 1)^3, x - 1, x - 1$.

Eigenvalue $\pm i$: Largest block size $n_1 = 2$. Algebraic multiplicity is 3, i.e., $2 + n_2 = 3$, i.e., $n_2 = 1$. Thus for $\lambda = i$ ($-i$ is analogous), $J_2(i) \oplus J_1(i)$.

For $\pm i$ together, the elementary divisors are $(x^2 + 1)^2, x^2 + 1$.

The invariant factors are: $(x - 1)^3(x^2 + 1)^2, (x - 1)(x^2 + 1), x - 1$.

2. Let A be a normal matrix. Prove that A^* commutes with any matrix that commutes with A .

It was shown in Theorem 8.1 that A normal implies there exists a polynomial $f(x)$ such that $A^* = f(A)$. If $AB = BA$, then $A^*B = f(A)B = Bf(A) = BA^*$.

3. (a) Assume $A \in F^{n \times n}$ is idempotent. Prove that the rank of A is equal to the number of nonzero eigenvalues of A .

Any idempotent matrix is similar to a diagonal matrix with 0s and 1s on the diagonal (because its minimal polynomial divides $x(x - 1)$). The rank of such a diagonal matrix is the number of nonzero eigenvalues, and both rank and spectrum are invariant under similarity.

- (b) Give a counterexample to show that the statement in 3a is false without the assumption that A is idempotent.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

4. Find the reduced singular value decomposition (i.e., find singular values and right

and left singular vectors for nonzero singular values) for $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$A^*A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$. $p_A(x) = x^2 - 13x + 36 = (x - 9)(x - 4)$, so $\sigma_1 = 3, \sigma_2 = 2$ are the singular values.

Find right singular vectors: $A^*A - 9I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$, so $\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$A^*A - 4I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, so $\mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Find left singular vectors (for σ_1, σ_2): $\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$.

$\mathbf{u}_2 = \frac{1}{\sigma_2} A \mathbf{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

5. Let $A, B \in \mathbb{C}^{n \times n}$. Prove that $\lambda_{\max}(A - B) + \lambda_{\min}(B) \leq \lambda_{\max}(A)$.

It is a consequence of Rayleigh-Ritz (also Theorem 7.10 in Zhang) that

$$\lambda_{\max}(C + G) \leq \lambda_{\max}(C) + \lambda_{\max}(G).$$

Let $C = A, G = -B$, so $\lambda_{\max}(G) = \lambda_{\max}(-B) = -\lambda_{\min}(B)$. Thus

$$\lambda_{\max}(A - B) \leq \lambda_{\max}(A) - \lambda_{\min}(B)$$

and the result follows.

6. Let $A \in \mathbb{C}^{n \times n}$ and $\lambda \in \mathbb{C}$. Prove that

$$\text{rank}((A - \lambda I)^2) + \text{rank}((A - \lambda I)^8) \geq \text{rank}((A - \lambda I)^5) + \text{rank}((A - \lambda I)^5).$$

Proof. Let $\#(m) =$ the number of Jordan blocks of size m with eigenvalue λ . Then

$$\text{rank}((A - \lambda I)^k) =$$

$$n - 1\#(1) - 2\#(2) - \dots - (k-1)\#(k-1) - k\#(k) - k\#(k+1) - \dots - k\#(n).$$

$$\text{rank}((A - \lambda I)^2) + \text{rank}((A - \lambda I)^6) =$$

$$\begin{aligned} & n - 1\#(1) - 2\#(2) - 2\#(3) - 2\#(4) - 2\#(5) - 2\#(6) - 2\#(7) - \dots \\ & + n - 1\#(1) - 2\#(2) - 3\#(3) - 4\#(4) - 5\#(5) - 6\#(6) - 6\#(7) - \dots \\ & = 2n - 2\#(1) - 4\#(2) - 5\#(3) - 6\#(4) - 7\#(5) - 8\#(6) - 8\#(7) - \dots \end{aligned}$$

whereas

$$\text{rank}((A - \lambda I)^4) + \text{rank}((A - \lambda I)^4) = 2\text{rank}((A - \lambda I)^4)$$

$$\begin{aligned} & 2(n - 1\#(1) - 2\#(2) - 3\#(3) - 4\#(4) - 4\#(5) - 4\#(6) - 4\#(7) - \dots) \\ & = 2n - 2\#(1) - 4\#(2) - 6\#(3) - 8\#(4) - 8\#(5) - 8\#(6) - 8\#(7) - \dots \end{aligned}$$

□

7. Let $A \in F^{n \times n}$ and let $\lambda \in \sigma(A)$. Prove that A is similar to a matrix having the sum of the elements in each row equal to λ .

Proof. Let \mathbf{v} be an eigenvector for A for eigenvalue λ . Let $\mathbf{1}$ be the all-ones vector. Extend to bases $\mathcal{B} = \{\mathbf{v}_1 = \mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1 = \mathbf{1}, \mathbf{w}_2, \dots, \mathbf{w}_n\}$. Let S be the transition matrix from \mathcal{C} to \mathcal{B} , i.e., for $i = 1, \dots, n$, $S\mathbf{w}_i = \mathbf{v}_i$. In particular, $S\mathbf{1} = \mathbf{v}$. Then $S^{-1}AS\mathbf{1} = S^{-1}A\mathbf{v} = S^{-1}\lambda\mathbf{v} = \lambda S^{-1}\mathbf{v} = \lambda\mathbf{1}$, so the sum of each row of $S^{-1}AS$ is λ . \square

8. Let $H \in \mathbb{C}^{n \times n}$ be Hermitian and partition H as $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ with H_{11}, H_{22} square. Prove that the Schur complement H/H_{11} is Hermitian.

Since H is Hermitian, $H_{11}^* = H_{11}$, $H_{22}^* = H_{22}$ and $H_{21} = H_{12}^*$.

$$H/H_{11} = H_{22} - H_{21}H_{11}^{-1}H_{12} = H_{22} - H_{12}^*H_{11}^{-1}H_{12}.$$

So $(H/H_{11})^* = H/H_{11}$ and H/H_{11} is Hermitian.