May 4, 2004 L. Hogben Math 301 Final Name:________________________

Directions Write your answers on your own paper. Answer 10 of the 12 questions (you may answer all questions but only the10 best scores will be used). Each question is worth the same amount. Every permutation must be expressed as a product of disjoint cycles unless otherwise specified. Every element of Z_n or $U(n)$ must be expressed as $[k]$ with $0 \leq k \leq n$. The following standard notation is also used: Z = integers and Z_n = integers mod n, both as a group under addition or as a ring; $U(n)$ is the multiplicative group of units of Z_n, ; $Q = \{1, i. j, k, -1, -i, -j, -k\}$; S_n is the group of permutations on $\{1,\ldots,n\}$.

1. In S_6, let $\alpha = (1\ 4)(1\ 3)(1\ 2)(3\ 6\ 5)(3\ 5)$
   a) $|\alpha| =$ b) $\alpha^{-1} =$

2. For each pair of groups G and H, determine whether G and H are isomorphic
   a) $G = Z_8 \oplus Z_{20} \oplus Z_{15}$ and $H = Z_{100} \oplus Z_{24}$
   b) $G = Z_6 \oplus Z_{20}$ and $H = Z_{12} \oplus Z_{10}$

3. In $U(21)$, let $H = \{[1], [4], [16]\}$ (assume H is a subgroup of $U(21)$). List all the distinct left cosets of H and for each coset, list the elements in the coset.

4. In Q, let $N = \{1, -1\}$. Assume the following information:
   N is a normal subgroup of Q.
   The distinct left cosets of N are $1N = \{1,-1\}$, $iN = \{i, -i\}$, $jN = \{j, -j\}$, $kN = \{k, -k\}$.
   Using the names listed above, make a Cayley Table for Q/N.

5. Let $R = Z \oplus Z$ and $I = \{(2a,b) \mid a,b \in Z\}$
   Show I is an ideal of R.

6. For each pair of groups G and H, give a reason that H cannot be a homomorphic image of G.
   a) $G = Z_8 \oplus Z_3$, $H = A_4$
   b) $G = Q \oplus U(5)$, $H = Z_5$

7. Give examples of
   a) A ring that is not commutative.
   b) A ring that has nonzero elements a and b with $ab = 0$.

8. Let G be a group and let H be a subgroup of G. Define $N(H) = \{x \in G \mid xH = Hx\}$. Show $N(H)$ is a subgroup of G.

9. Let G be a group such that for all $x, y \in G$, $(xy)^2 = x^2 y^2$. Prove that G is abelian.

10. Prove that for any finite group G and any homomorphism φ, $|\varphi(G)|$ divides $|G|$. [Note: This is a theorem in the text. Needless to say, you cannot cite that theorem in your proof.]

11. For a finite group, the exponent of a group is defined to be the least positive integer m such that for all $x \in G$, $x^m = e$. Prove that for any finite group G with exponent m, m divides $|G|$.

12. Let R be a ring and let $a \in R$ be such that $a^2 = 0$. Prove that for any x in R, a commutes with $xa + ax$.