Farey Symbols and Subgroups of $\mathrm{SL}_2(\mathbb{Z})$

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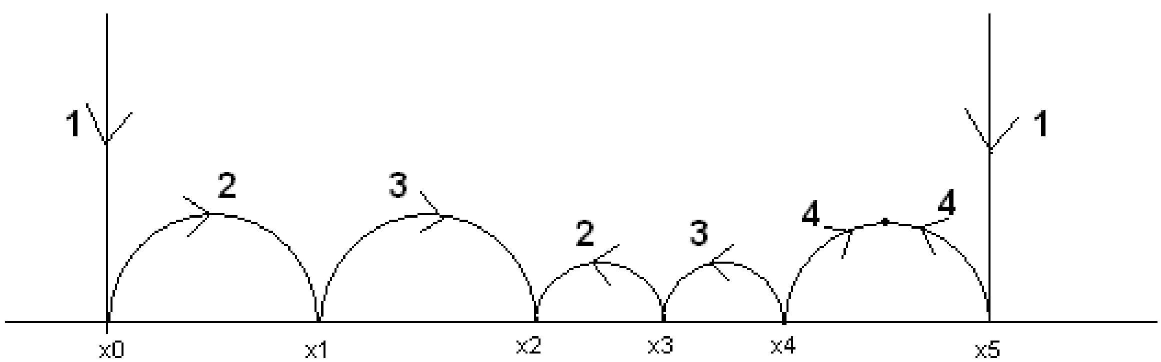


$\mathrm{SL}_2(\mathbb{Z})$

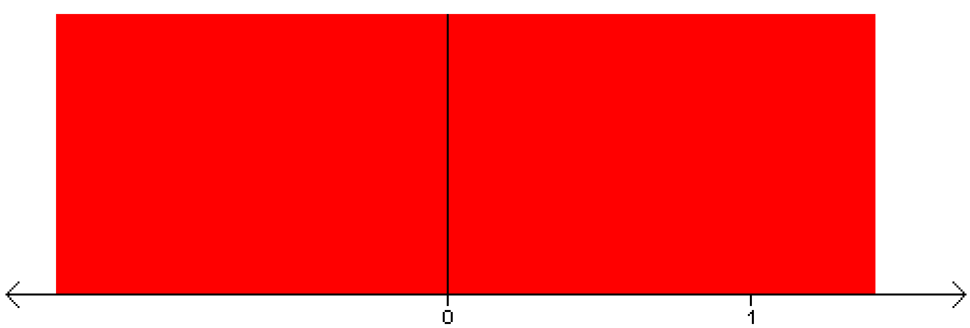
$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}$$

$\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C} \cup \{\infty\}$ as follows:

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\gamma z = \frac{az+b}{cz+d}$

γ and $-\gamma$ act in the same way, so we work with $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I, -I\}$

$\mathbb{R} \cup \{\infty\}$ is invariant under every γ, so in fact $\mathrm{PSL}_2(\mathbb{Z})$ acts on $H = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$, the upper half plane

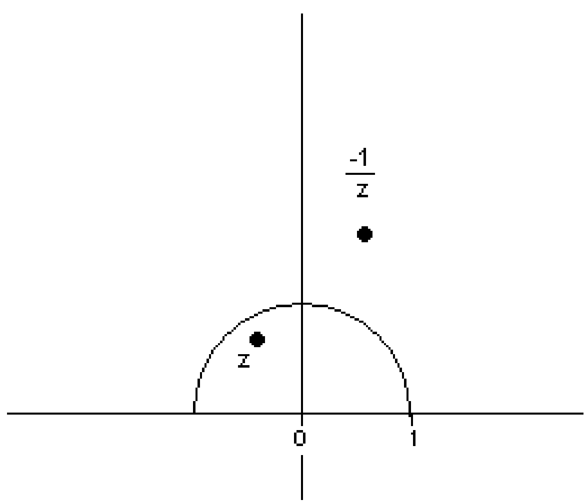


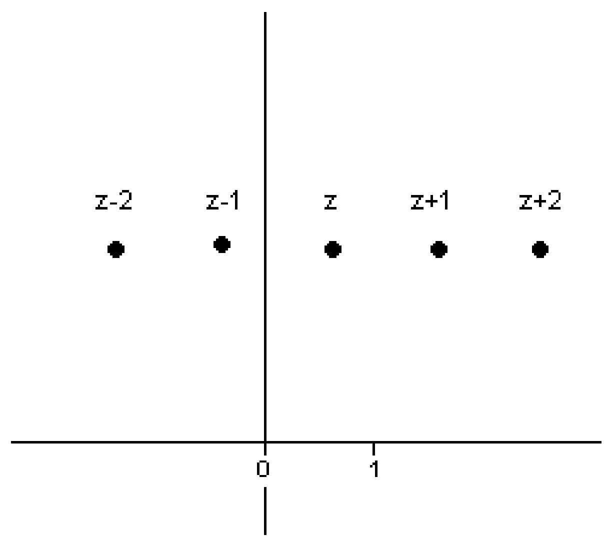
Generators for $\mathrm{PSL}_2(\mathbb{Z})$

$\mathrm{PSL}_2(\mathbb{Z})$ is generated by two elements:

- $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ corresponding to $z \mapsto \frac{-1}{z}$
- $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ corresponding to $z \mapsto z + 1$

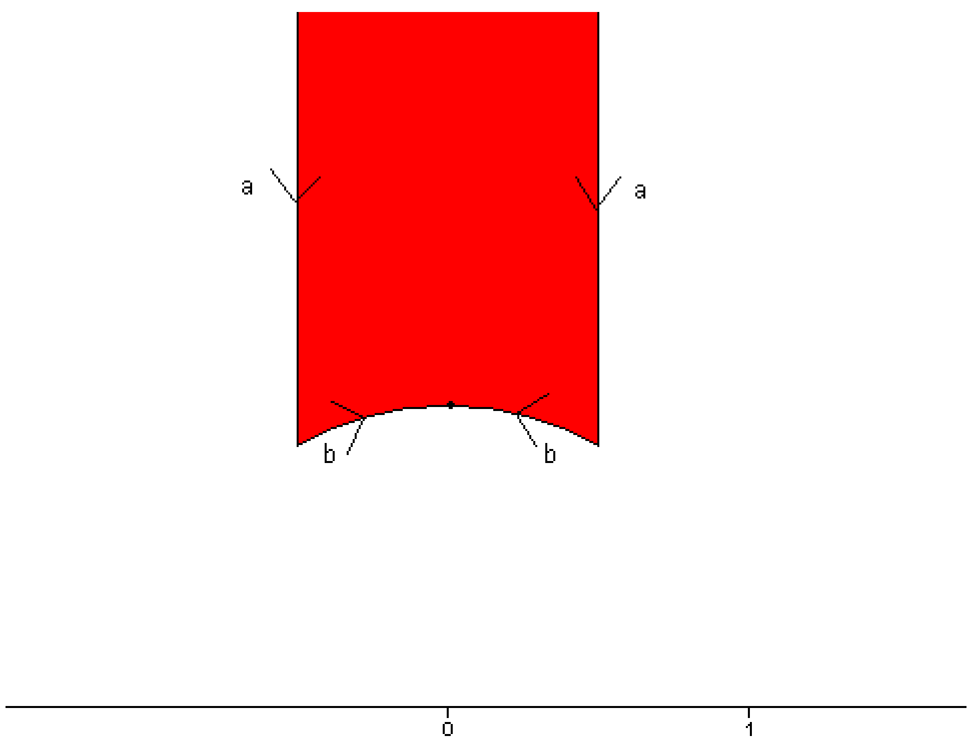
The Generating Maps:





Fundamental Domain

$F = \{z \in H : -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2}$ and $|z| \geq 1\}$ is called the **fundamental domain** of $\mathrm{PSL}_2(\mathbb{Z})$



$z \sim w$ if $z = \gamma w$ for some γ in $\mathrm{PSL}_2(\mathbb{Z})$

Every point of H is equivalent to exactly one point in H (or maybe two boundary points).

Modular Functions

Modular Functions of $\mathrm{PSL}_2(\mathbb{Z})$ are meromorphic functions $f(z)$ such that $f(\gamma z) = f(z)$ for every $z \in H, \gamma \in \mathrm{PSL}_2(\mathbb{Z})$

Equivalently: functions $f(z)$ such that:

- $f(z+1) = f(z)$
- $f(-1/z) = f(z)$

Subgroups of $\mathrm{PSL}_2(\mathbb{Z})$

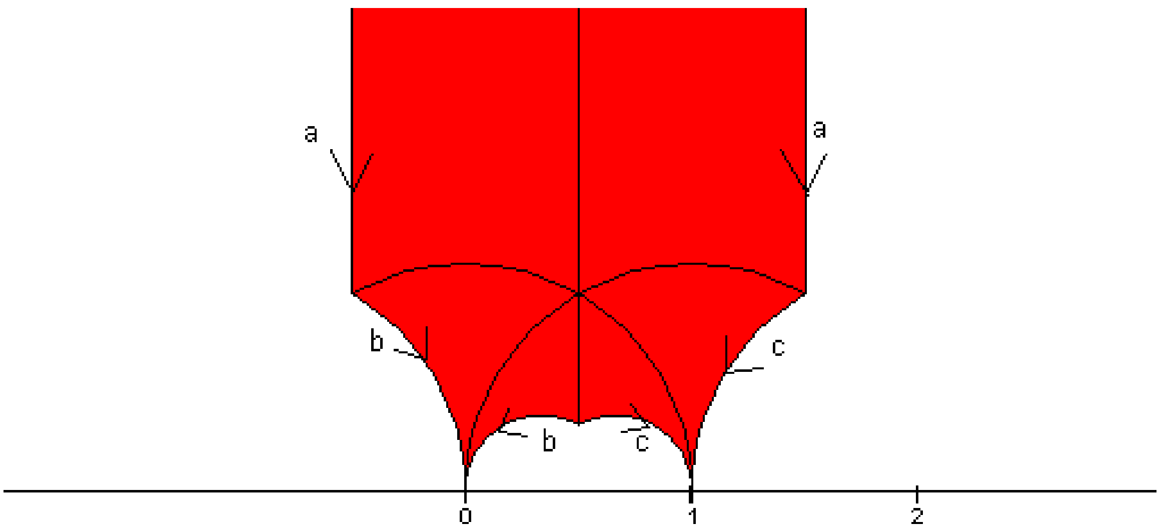
For positive integers N definte subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ by:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}$$

$$\Gamma^1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod N \right\}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \mod N \right\}$$

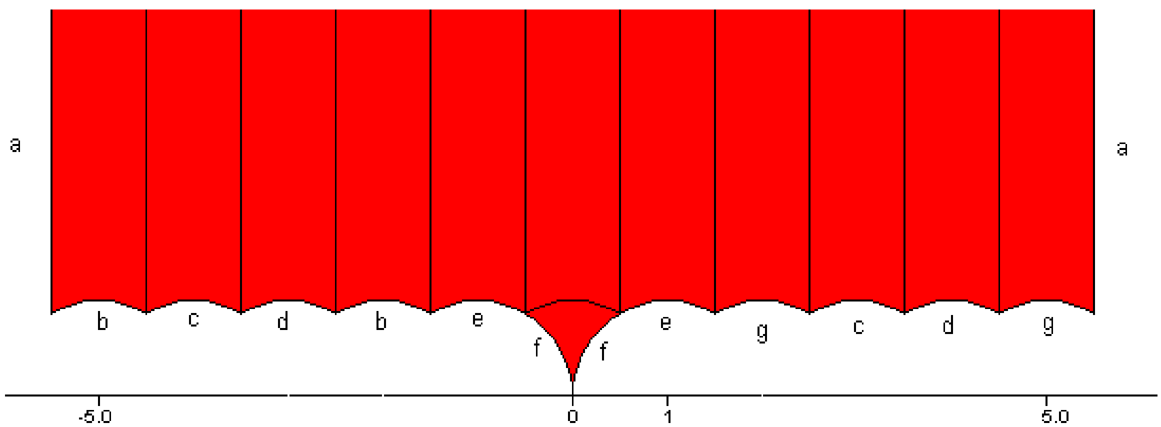
Example: $\Gamma(2)$



Coset Representatives:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$$

Example: $\Gamma^0(11)$



Coset Representatives:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad -5 \leq k \leq 5$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Congruence/Noncongruence Groups

$\Gamma(N)$, $\Gamma^0(N)$ and $\Gamma^1(N)$ are examples of congruence groups. Γ is congruence if it contains $\Gamma(N)$ for some N.

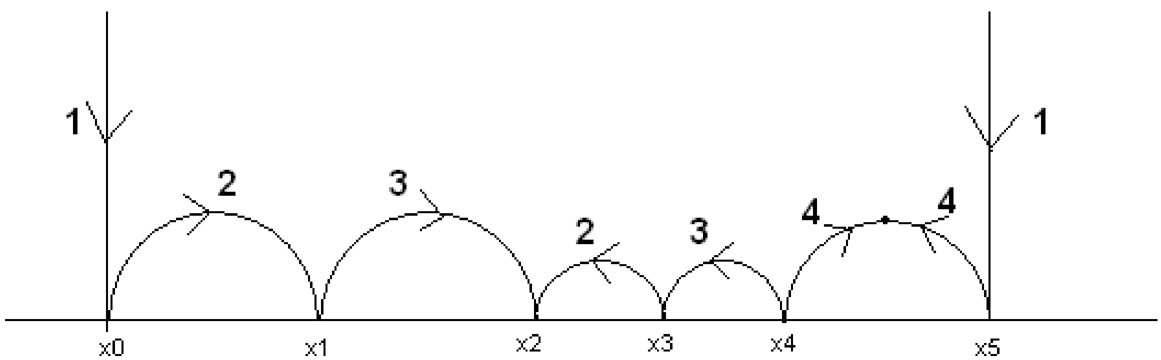
Congruence groups are well-studied and well-understood. Noncongruence groups not so much.

Farey Symbols

Special Polygon: A special kind of fundamental domain that touches $\mathbb{R} \cup \{\infty\}$ at a sequence of reduced rational numbers:

$$\infty = \frac{-1}{0}, \frac{a_0}{b_0}, \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}, \frac{1}{0} = \infty$$

satisfying $a_i b_{i-1} - a_{i-1} b_i = 1$

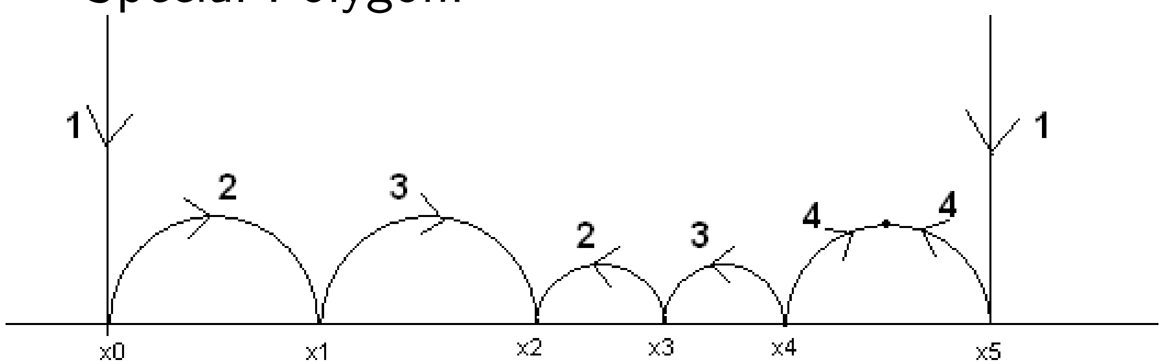


The sequence is called a *Generalized Farey sequence*

Farey Symbols (ctd.)

Farey Symbol: This sequence together with pairing information:

Special Polygon:

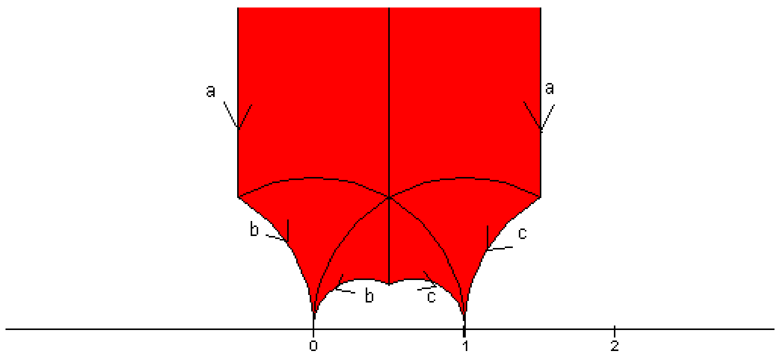


Farey Symbol:

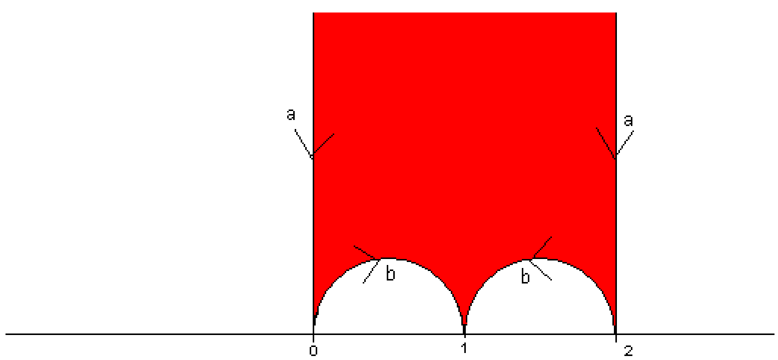
$$\infty \underbrace{\quad}_{1} x_0 \underbrace{\quad}_{2} x_1 \underbrace{\quad}_{3} x_2 \underbrace{\quad}_{2} x_3 \underbrace{\quad}_{3} x_4 \underbrace{\quad}_{\bigcirc} x_5 \underbrace{\quad}_{1} \infty$$

Example: $\Gamma(2)$

Standard Fundamental Domain:



Special Polygon:



Farey Symbol for $\Gamma(2)$.

$$\infty \underbrace{\quad}_{1} 0 \underbrace{\quad}_{2} 1 \underbrace{\quad}_{2} 2 \underbrace{\quad}_{1} \infty$$

Generators

For a fundamental domain of a group, the unique elements of $\mathrm{PSL}_2(\mathbb{Z})$ that map each side to its pair generate the group.

Example: $\Gamma(2)$

- Generators from Standard Fundamental domain:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

- Generators from Standard Fundamental domain:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

Note $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

Character Groups

Let Γ be a finite-index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$

$\Gamma = \langle \eta, \gamma_1, \ldots, \gamma_n \rangle$

φ: a homomorphism from Γ to a cyclic group where $\varphi(\eta) = \mu_n$, $\varphi(\gamma_i) = 1$ for all i

$\ker \varphi$ is a character group.

Character groups are usually noncongruence (even if Γ is not), and in this case the Farey Symbol can be easily calculted from that of Γ.