Math 165 (Chris Kurth)
Spring 2008
Quiz 4

Show your work. Answers without work will not receive credit.

1. (8 points) Evaluate $\int (x^4 - 5x^3 + 3)dx$

   Solution: $\int (x^4 - 5x^3 + 3)dx = \frac{1}{5}x^5 - \frac{5}{4}x^4 + 3x + C$

2. (8 points) Evaluate $\int_{-2}^{-1} \frac{4}{x^4}\ dx$

   Solution:

$$\begin{aligned}\int_{-2}^{-1} \frac{4}{x^4}\ dx &= \int_{-2}^{-1} 4x^{-4}dx \\ &= -\frac{4}{3}x^{-3}\Big|_{-2}^{-1} \\ &= -\frac{4}{3}\left((-1)^{-3} - (-2)^{-3}\right) \\ &= \frac{7}{6}\end{aligned}$$

3. (10 points) Evaluate $\int_0^{\pi/2} \sqrt{1 - \sin x} \cos x\ dx$

   Solution: Let $u = 1 - \sin x$. Then $du = -\cos x dx$ and

$$\begin{aligned}\int_0^{\pi/2} \sqrt{1 - \sin x} \cos x\ dx &= \int_1^0 -u^{1/2}du \\ &= \frac{-2}{3}u^{3/2}\Big|_1^0 \\ &= \left(\frac{-2}{3} \cdot 0^{3/2} - \frac{-2}{3} \cdot 1^{3/2}\right) \\ &= \frac{2}{3}\end{aligned}$$

4. (8 points) Evaluate $\frac{d}{dx}\int_0^{x^4} \sec t\ dt$

   Solution: Let $f(u) = \frac{d}{dx}\int_0^{u} \sec t\ dt$ and $g(x) = x^4$. Then by the First

Fundamental Theorem of Calculus, $f'(u) = \sec(u)$, and by the chain rule:

$$\begin{aligned} \frac{d}{dx}\int_0^u \sec t \; dt &= \frac{d}{dx} f(g(x)) \\ &= f'(g(x))g'(x) \\ &= \sec(x^4)\cdot 4x^3 \end{aligned}$$

5. (12 points) Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$.

   Solution: The distance from (x, y) to $(2, 0)$ is $d = \sqrt{(x-2)^2 + y^2}$. Minimize the square of the distance, i.e. let $f(x) = d^2 = (x-2)^2 + y^2 = (x-2)^2 + x^2 + 4 = 2x^2 - 4x + 8$ and minimize $f(x)$.

   The critical point is where $0 = f'(x) = 4x - 4$, i.e. $x = 1$, and $y^2 = x^2 + 4 = 5$. So the closest points are $(1, \pm\sqrt{5})$.

6. (12 points) Find the local extreme values of $f(x) = x^3 - \frac{15}{2}x^2 + 18x - 7$ and determine if each is a local maximum, minimum or neither.

   Solution: Note $f'(x) = 3x^2 - 15x + 18 = 3(x^2 - 5x + 6) = 3(x-2)(x-3)$, and $f''(x) = 6x - 15$. So $f'(x) = 0$ when $x = 2$ or $x = 3$ and $f''(2) = -3 < 0$ and $f''(3) = 3 > 0$. Thus by the second derivative test, there is a local maximun at $x = 2$ and a local minimum at $x = 3$.

7. (12 points) Find the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x}{y\sqrt{1+x^2}}$ such that $y = 2$ when $x = 0$.

   Solution: $y \; dy = \dfrac{x}{\sqrt{1+x^2}}dx$, so we integrate $\displaystyle\int y \; dy = \int \frac{x}{\sqrt{1+x^2}}dx$. Let $u = 1 + x^2$, so $du = 2xdx$. and

$$\frac{1}{2}y^2 = \frac{1}{2}\int u^{-1/2}du + C = u^{1/2} + C = \sqrt{1+x^2} + C$$

   At $(2, 0)$ this is $2 = 1 + C$, i.e. $C = 1$. Plug in C to get:

$$\frac{1}{2}y^2 = \sqrt{1+x^2} + 1$$

8. (10 points) Sketch the graph of $f(x) = \dfrac{x+2}{x-1}$. Indicate where the function is increasing and decreasing and where the asymptotes are.

   Solution: $f'(x) = \dfrac{-3}{(x-2)^2} < 0$ so $f(x)$ is always decreasing. It has a horizontal asymptote $y = 1$ (since $\lim_{x\to\infty} f(x) = 1$) and a vertical asymptote $x = 1$ (since $\lim_{x\to 1} f(x)$ goes to infinity).

9. (12 points) Use the definition of the definite integral (i.e. Riemann sums) to calculate:

$$\int_0^3 x^2 dx$$

Solution: Let $\Delta x = \frac{3}{n}$, $x_i = i \cdot \frac{3}{n}$, and $\overline{x_i} = x_i$. Then:

$$\begin{aligned}
\int_0^3 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(i \cdot \frac{3}{n}\right)^2 \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{9}{2} \lim_{n\to\infty} \frac{2n^3 + \dots}{n^3} \\
&= 9
\end{aligned}$$

10. (8 points) Evaluate $\int_{-\pi/4}^{\pi/4} x \sin^2(x)\ dx$

Solution: $\int_{-\pi/4}^{\pi/4} x \sin^2(x)\ dx = 0$ since it is the integral of an odd function over a symmetric interval.