Math 166 D2 (Kurth)
HW #1

THE GAMMA FUNCTION (Due March 9)

The Gamma function, $\Gamma(t)$, is defined

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$$

where this integral converges.

1. Use this formula to calculate $\Gamma(1), \Gamma(2), \Gamma(3), \Gamma(4)$.

2. Show that $\Gamma(t+1) = t\Gamma(t)$ for every $t \neq 0, -1, -2, \ldots$
(Hint: Note that $\lim_{b \to \infty} \frac{b^t}{e^b} = 0$ for every t. We did this in class for integer t.)

3. What can we then conclude about the relation between $\Gamma(n)$ and $n! = 1 \cdot 2 \cdot \cdots \cdot n$, the factorial?

So $\Gamma(t)$ is a familiar function for positive integer values. But it also converges and is continuous for all real numbers t where $t \neq 0, -1, -2, \ldots$.

4. Calculate $\Gamma(\frac{1}{2})$. Use this value to find $\Gamma(\frac{3}{2})$, $\Gamma(\frac{5}{2})$ and $\Gamma(-\frac{1}{2})$.
(Hint: Recall that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} dx = 1$)

5. For $r > 0$ Let $B_n(r)$ be set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2$. (i.e. $B_1(r)$ is a line segment of length $2r$, $B_2(r)$ is a disc of radius r, etc.). Then the "n-volume" of $B_n(r)$ is:

$$\text{Vol}(B_n(r)) = \frac{2r^n \pi^{n/2}}{n\Gamma(n/2)}$$

(so here 1-volume would be length, 2-volume is area, 3-volume is ordinary volume, etc.). Verify the values of $\text{Vol}(B_1(r))$, $\text{Vol}(B_2(r))$, $\text{Vol}(B_3(r))$ from what you already know. What are $\text{Vol}(B_4(r))$ and $\text{Vol}(B_5(r))$?