Example 10.3 A split-plot experiment with whole plots arranged in blocks.

Blocks: $r = 4$ fields (or locations).

Whole plots:

Each field is divided into $a = 2$ whole plots.

Whole plot factor:

two cultivars of grasses (A, B)

- within each block, cultivar A is grown in one whole plot, cultivar B is grown in the other
- separate random assignments of cultivars to whole plots is done in each block

Sub-plot factor:

$b = 3$ bacterial innoculation treatments:

CON	for control (no innoculation)
DEA	for dead
LIV	for live

Each whole plot is split into three sub-plots and independent random assignments of innoculation treatments to sub-plots are done within whole plots.

Measured response:

Dry weight yield

Source:

Littel, R.C. Freund, R.J. and Spector, P.C. (1991) SAS Systems for Linear Models, 3rd edition, SAS Institute, Cary, NC

Data: grass.dat

SAS code: grass.sas

S-PLUS code: grass.ssc

Block 1

Cultivar B	CON	DEA	LIV
Cultivar A	LIV	CON	DEA

Block 2

DEA	LIV
LIV	CON
CON	DEA
Cultivar A	Cultivar B

Block 3

DEA	CON
CON	DEA
LIV	LIV
Cultivar B	Cultivar A

Block 4

LIV	DEA	CON	Cultivar B
CON	DEA	LIV	Cultivar A

Model with random block effects:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}$$

$$i = 1, \ldots, a$$
$$j = 1, \ldots, r$$
$$k = 1, \ldots, b$$

$Y_{ijk} \Rightarrow$ observed yield for the k-th innoculant applied to the i-th cultivar in the j-th field

$\alpha_i \Rightarrow$ fixed cultivar effect

$\gamma_k \Rightarrow$ fixed innoculant effect

$\delta_{ik} \Rightarrow$ cultivar*innoculant interaction

The following random effects are independent of each other:

$\beta_j \sim NID(0, \sigma_\beta^2) \Rightarrow$ random block effects

$\eta_{ij} \sim NID(0, \sigma_w^2) \Rightarrow$ random whole plot effects

$e_{ijk} \sim NID(0, \sigma_e^2) \Rightarrow$ random errors

ANOVA table:

Source of Variation	df	SS
Blocks	$r-1=3$	25.32
Cultivars	$a-1=1$	2.41
Whole Plot error (Block $\times$ cultivar interaction)	(r-1)(a-1)=3	9.48
Innoculants	$b-1=2$	118.18
Cult. $\times$ Innoc.	$(a-1)(b-1)=2$	1.83
Sub-plot error	12	8.465
Corrected total	23	165.673

ANOVA table:

Source of Variation	df	MS	$E(MS)$
Blocks	3	8.44	$\sigma_e^2 + b\sigma_w^2 + ba\sigma_\beta^2$
Cultivars	1	2.41	$\sigma_e^2 + b\sigma_w^2 + (i)$
Whole Plot error (Block $\times$ cult. interaction)	3	3.16	$\sigma_e^2 + b\sigma_w^2$
Innoculants	2	59.09	$\sigma_e^2 + (ii)$
Cult. $\times$ Innoc.	2	0.91	$\sigma_e^2 + (iii)$
Sub-plot error	12	0.705	σ_e^2
Corrected total	23		

(i) $\dfrac{br\Sigma_{i=1}^{a}(\alpha_i + \bar{\delta}_{i.} - \bar{\alpha}_{.} - \bar{\delta}_{..})^2}{a-1}$

(ii) $\dfrac{ar\Sigma_{k=1}^{b}(\gamma_k + \bar{\delta}_{.k} - \bar{\gamma}_{.} - \bar{\delta}_{..})^2}{b-1}$

(iii) $\dfrac{r\Sigma_i \Sigma_k(\delta_{ik} - \bar{\delta}_{1.} - \bar{\delta}_{.k} + \bar{\delta}_{..})^2}{(a-1)(b-1)}$

```
> # This file is posted as  grass.ssc

> #  Enter the data into a data frame and
> #  define factors

> grass <- read.table("c/st511/grass.dat",
+    col.names=c("Block","Culti","Innoc","Yield"))
> grass$Block <- as.factor(grass$Block)
> grass$Culti <- as.factor(grass$Culti)
> grass$Innoc <- as.factor(grass$Innoc)
> grass

   Block Culti Innoc Yield
 1     1     A   CON  27.4
 2     1     A   DEA  29.7
 3     1     A   LIV  34.5
 4     1     B   CON  29.4
 5     1     B   DEA  32.5
 6     1     B   LIV  34.4
 7     2     A   CON  28.9
 8     2     A   DEA  28.7
 9     2     A   LIV  33.4
10     2     B   CON  28.7
```



```
11     2     B   DEA  32.4
12     2     B   LIV  36.4
13     3     A   CON  28.6
14     3     A   DEA  29.7
15     3     A   LIV  32.9
16     3     B   CON  27.2
17     3     B   DEA  29.1
18     3     B   LIV  32.6
19     4     A   CON  26.7
20     4     A   DEA  28.9
21     4     A   LIV  31.8
22     4     B   CON  26.8
23     4     B   DEA  28.6
24     4     B   LIV  30.7
```



```
> # Multistratum models may be fitted using aov() if
> # the design is balanced and all factors are random.
> # Models are specified by a model formula
> #
> #  response ~ mean.formula + Error(strata.formular)

> summary(grass.aov <- aov(Yield ~ Culti*Innoc +
          Error(Block/Culti),data=grass))


Error: Block
          Df Sum of Sq Mean Sq F Value Pr(F)
Residuals  3     25.32    8.44

Error: Culti %in% Block
          Df Sum of Sq  Mean Sq   F Value     Pr(F)
    Culti  1  2.406667 2.406667 0.7616034 0.4470549
Residuals  3  9.480000 3.160000

Error: Within
            Df Sum of Sq  Mean Sq  F Value     Pr(F)
      Innoc  2  118.1758 59.08792 83.76314 0.0000001
Culti:Innoc  2    1.8258  0.91292  1.29415 0.3097837
  Residuals 12    8.4650  0.70542
```



```
> #Use the lme function to fit the split-plot
> # model with random block effects.  Here
> # each cultivar is used only once in each
> # block, so cultivars within blocks can be
> # used to denote the whole plots within
> # blocks.  The code  Block/Culti  in the
> # random statement designates random
> # block effects and random effects for
> # cultivars within blocks (whole plots),

> options(contrasts=c(factor="contr.treatment",
+                    ordered="contr.poly"))
> grass.lme <- lme( Yield ~ Culti*Innoc ,
+    data=grass,
+    random = ~ 1 | Block/Culti )

> # Print F-tests for fixed effects
> anova(grass.lme)

            numDF denDF  F-value p-value
(Intercept)     1    12 2630.822  <.0001
      Culti     1     3    0.762  0.4471
      Innoc     2    12   83.763  <.0001
Culti:Innoc     2    12    1.294  0.3098
```

```
> # Print estimates of variance components
> VarCorr(grass.lme)

             Variance    StdDev
    Block = pdSymm(~ 1)
(Intercept) 0.8800030   0.9380847
    Culti = pdSymm(~ 1)
(Intercept) 0.8181924   0.9045399
   Residual 0.7054168   0.8398909


> # Print estimates of fixed effects
> fixef(grass.lme)

 (Intercept)   Culti   InnocDEA   InnocLIV
        27.9   0.125       1.35       5.25

          CultiInnocDEA   CultiInnocLIV
                  1.275            0.25
```



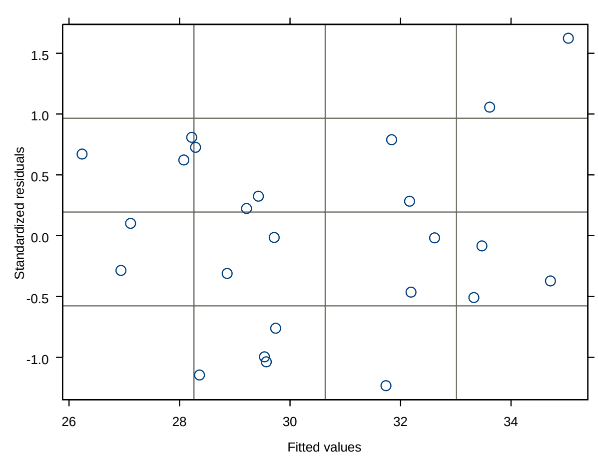
```
> # Print predictions of random effects
> ranef(grass.lme)

Level: Block
  (Intercept)
1   0.5630343
2   0.6255937
3  -0.2502375
4  -0.9383905

Level: Culti %in% Block
    (Intercept)
1/A -0.10074734
1/B  0.62423465
2/A -0.30469464
2/B  0.88634720
3/A  0.42740598
3/B -0.66006701
4/A -0.02196399
4/B -0.85051484
```



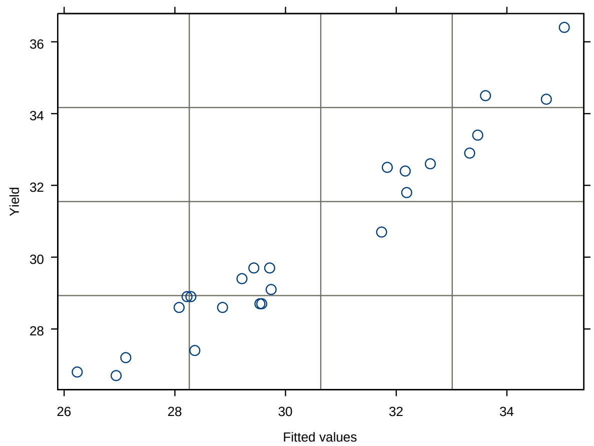
```
> # Print approximate confidence intervals
> intervals(grass.lme)

Approximate 95% confidence intervals

 Fixed effects:
                    lower   est.     upper
  (Intercept)  26.21102928 27.900 29.588971
        Culti  -2.65268424  0.125  2.902684
     InnocDEA   0.05601926  1.350  2.643981
     InnocLIV   3.95601926  5.250  6.543981
CultiInnocDEA  -0.55496511  1.275  3.104965
CultiInnocLIV  -1.57996511  0.250  2.079965

 Random Effects:
  Level: Block
                   lower      est.    upper
sd((Intercept)) 0.1370272 0.9380847 6.422103
  Level: Culti
                   lower      est.    upper
sd((Intercept)) 0.2077634 0.9045399 3.938096

 Within-group standard error:
     lower      est.    upper
 0.4772925 0.8398909 1.477955
```



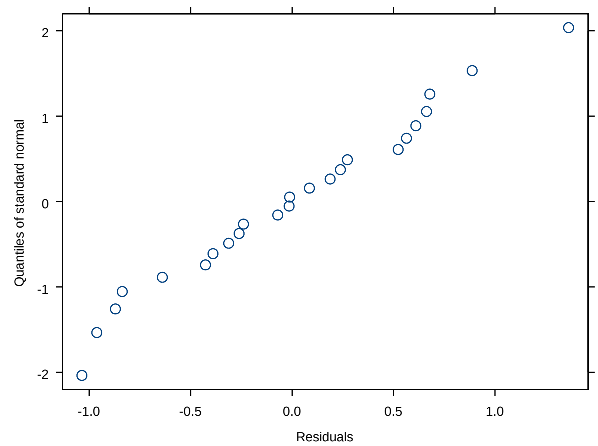
```
> #Plot residuals versus fitted values
> par(fin=c(6,6),cex=1.5,mkh=1.2,mex=1.5)
> plot(grass.lme, resid(.,type="p") ~ fitted(.),
+       id=0.01, adj=-0.3)
```



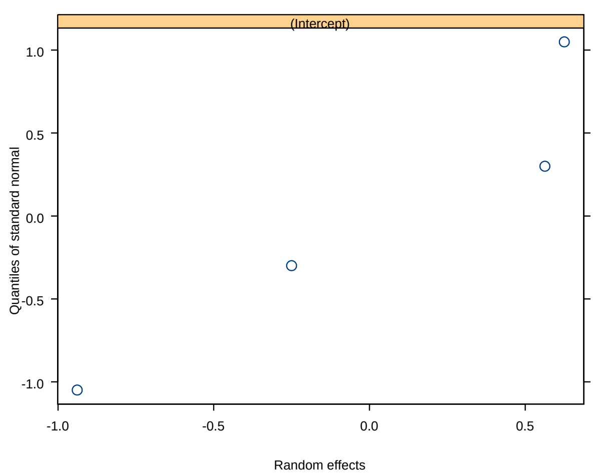
```
> # Plot observations versus fitted values
> plot(grass.lme, Yield ~ fitted(.),
+        id=0.01, adj=-0.3)
```



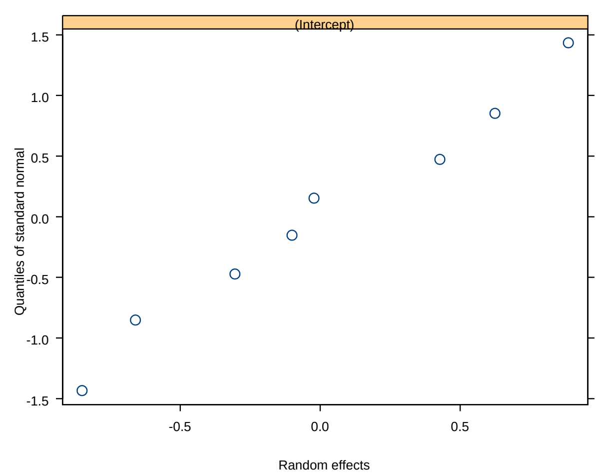

```
> # Normal probability plot of within whole
> # plot residuals
> qqnorm( grass.lme, ~resid(.))
```




```
> # Normal probability plot of random
> # effects
> qqnorm( grass.lme, ~ranef(.,level=1), id=.01)
```




```
> qqnorm( grass.lme, ~ranef(.,level=2), id=.01)
```



```
/*  SAS code for analyzing the data from the
    split plot experiment corresponding to
    example 10.3 in class   */


data set1;
  infile 'grass.dat';
  input block cultivar $ innoc $ yield;
  run;


proc print data=set1;  run;


proc mixed data=set1;
  class block cultivar innoc;
  model yield = cultivar innoc cultivar*innoc
                 / ddfm=satterth;
  random block block*cultivar;
  lsmeans cultivar / e  pdiff tdiff;
  lsmeans innoc / e pdiff tdiff;
  lsmeans cultivar*innoc / e pdiff tdiff;
  estimate  'a:live vs b:live'
               cultivar 1 -1 innoc 0 0 0
               cultivar*innoc 0 0 1 0 0 -1;
  estimate  'a:live vs b:live'
               cultivar 1 -1 innoc 0 0 1
               cultivar*innoc 0 0 1 0 0 -1 / e;
  estimate  'a:live vs a:dead'
               cultivar 0 0 innoc 0 -1 1
               cultivar*innoc 0 -1 1 0 0 0 / e;
run;
```



```
/*  Use Proc GLM to compute an ANOVA table */

proc glm data=set1;
  class block cultivar innoc;
  model yield = cultivar block cultivar*block
              innoc cultivar*innoc;
  random block block*cultivar;
  test h=cultivar block e=cultivar*block;
run;
```


OBS	BLOCK	CULTIVAR	INNOC	YIELD
1	1	A	CON	27.4
2	1	A	DEA	29.7
3	1	A	LIV	34.5
4	1	B	CON	29.4
5	1	B	DEA	32.5
6	1	B	LIV	34.4
7	2	A	CON	28.9
8	2	A	DEA	28.7
9	2	A	LIV	33.4
10	2	B	CON	28.7
11	2	B	DEA	32.4
12	2	B	LIV	36.4
13	3	A	CON	28.6
14	3	A	DEA	29.7
15	3	A	LIV	32.9
16	3	B	CON	27.2
17	3	B	DEA	29.1
18	3	B	LIV	32.6
19	4	A	CON	26.7
20	4	A	DEA	28.9
21	4	A	LIV	31.8
22	4	B	CON	26.8
23	4	B	DEA	28.6
24	4	B	LIV	30.7

The MIXED Procedure

Class Level Information

Class	Levels	Values
BLOCK	4	1 2 3 4
CULTIVAR	2	A B
INNOC	3	CON DEA LIV

REML Estimation Iteration History

Iteration	Evaluations	Objective	Criterion
0	1	42.10326642	
1	1	31.98082956	0.00000000

Convergence criteria met.

Covariance Parameter Estimates (REML)

Cov Parm	Ratio	Estimate	Std Error	Z	Pr > |Z|
BLOCK	1.24749	0.88000	1.22640	0.72	0.4730
BLOCK*CULT	1.15987	0.81819	0.86538	0.95	0.3444
Residual	1.00000	0.70542	0.28799	2.45	0.0143

Model Fitting Information for YIELD

Description	Value
Observations	24.0000
Variance Estimate	0.7054
Standard Deviation Estimate	0.8399
REML Log Likelihood	-32.5313
Akaike's Information Criterion	-35.5313
Schwarz's Bayesian Criterion	-36.8669
-2 REML Log Likelihood	65.0626

Tests of Fixed Effects

Source	NDF	DDF	Type III F	Pr > F
CULTIVAR	1	3	0.76	0.4471
INNOC	2	12	83.76	0.0001
CULTIVAR*INNOC	2	12	1.29	0.3098

Coefficients for a:live vs b:live

Parameter	Row 1
INTERCEPT	0
CULTIVAR A	1
CULTIVAR B	-1
INNOC CON	0
INNOC DEA	0
INNOC LIV	1
CULTIVAR*INNOC A CON	0
CULTIVAR*INNOC A DEA	0
CULTIVAR*INNOC A LIV	1
CULTIVAR*INNOC B CON	0
CULTIVAR*INNOC B DEA	0
CULTIVAR*INNOC B LIV	-1

Coefficients for a:live vs a:dead

Parameter	Row 1
INTERCEPT	0
CULTIVAR A	0
CULTIVAR B	0
INNOC CON	0
INNOC DEA	-1
INNOC LIV	1
CULTIVAR*INNOC A CON	0
CULTIVAR*INNOC A DEA	-1
CULTIVAR*INNOC A LIV	1
CULTIVAR*INNOC B CON	0
CULTIVAR*INNOC B DEA	0
CULTIVAR*INNOC B LIV	0

ESTIMATE Statement Results

Parameter	Estimate	Std Error	DDF	T	Pr > |T|
a:live vs b:live	-0.375	0.87281	5.98	-0.43	0.6825
a:live vs b:live	.	.	.	.	.
a:live vs a:dead	3.900	0.59389	12	6.57	0.0001

Coefficients for CULTIVAR Least Squares Means

Parameter	Row 1	Row 2
INTERCEPT	1	1
CULTIVAR A	1	0
CULTIVAR B	0	1
INNOC CON	0.33333	0.33333
INNOC DEA	0.33333	0.33333
INNOC LIV	0.33333	0.33333
CULTIVAR*INNOC A CON	0.33333	0
CULTIVAR*INNOC A DEA	0.33333	0
CULTIVAR*INNOC A LIV	0.33333	0
CULTIVAR*INNOC B CON	0	0.33333
CULTIVAR*INNOC B DEA	0	0.33333
CULTIVAR*INNOC B LIV	0	0.33333

Coefficients for INNOC Least Squares Means

Parameter	Row 1	Row 2	Row 3
INTERCEPT	1	1	1
CULTIVAR A	0.5	0.5	0.5
CULTIVAR B	0.5	0.5	0.5
INNOC CON	1	0	0
INNOC DEA	0	1	0
INNOC LIV	0	0	1
CULTIVAR*INNOC A CON	0.5	0	0
CULTIVAR*INNOC A DEA	0	0.5	0
CULTIVAR*INNOC A LIV	0	0	0.5
CULTIVAR*INNOC B CON	0.5	0	0
CULTIVAR*INNOC B DEA	0	0.5	0
CULTIVAR*INNOC B LIV	0	0	0.5

Coefficients for CULTIVAR*INNOC Least Squares Means

Parameter	Row 1	Row 2	Row 3	Row 4	Row 5	Row 6
INTERCEPT	1	1	1	1	1	1
CULTIVAR A	1	1	1	0	0	0
CULTIVAR B	0	0	0	1	1	1
INNOC CON	1	0	0	1	0	0
INNOC DEA	0	1	0	0	1	0
INNOC LIV	0	0	1	0	0	1
CULTIVAR*INNOC A CON	1	0	0	0	0	0
CULTIVAR*INNOC A DEA	0	1	0	0	0	0
CULTIVAR*INNOC A LIV	0	0	1	0	0	0
CULTIVAR*INNOC B CON	0	0	0	1	0	0
CULTIVAR*INNOC B DEA	0	0	0	0	1	0
CULTIVAR*INNOC B LIV	0	0	0	0	0	1

Least Squares Means

Level	LSMEAN	Std. Error	DDF	T	Pr > |T|
CULTIVAR A	30.1000	0.6952	4.97	43.30	0.0001
CULTIVAR B	30.7333	0.6952	4.97	44.21	0.0001
INNOC CON	27.9625	0.6407	4.06	43.65	0.0001
INNOC DEA	29.9500	0.6407	4.06	46.75	0.0001
INNOC LIV	33.3375	0.6407	4.06	52.04	0.0001
CULTIVAR*INNOC A CON	27.9000	0.7752	7.5	35.99	0.0001
CULTIVAR*INNOC A DEA	29.2500	0.7752	7.5	37.73	0.0001
CULTIVAR*INNOC A LIV	33.1500	0.7752	7.5	42.76	0.0001
CULTIVAR*INNOC B CON	28.0250	0.7752	7.5	36.15	0.0001
CULTIVAR*INNOC B DEA	30.6500	0.7752	7.5	39.54	0.0001
CULTIVAR*INNOC B LIV	33.5250	0.7752	7.5	43.25	0.0001

Differences of Least Squares Means

Level 1	Level 2	Differ -rence	Std. Error	DDF	T	Pr > |T|
CULT A	CULT B	-0.6333	0.72571	3	-0.87	0.4471
IN CON	IN DEA	-1.9875	0.41994	12	-4.73	0.0005
IN CON	IN LIV	-5.3750	0.41994	12	-12.80	0.0001
IN DEA	IN LIV	-3.3875	0.41994	12	-8.07	0.0001
A CON	A DEA	-1.3500	0.59389	12	-2.27	0.0422
A CON	A LIV	-5.2500	0.59389	12	-8.84	0.0001
A CON	B CON	-0.1250	0.87281	5.98	-0.14	0.8908
A CON	B DEA	-2.7500	0.87281	5.98	-3.15	0.0199
A CON	B LIV	-5.6250	0.87281	5.98	-6.44	0.0007
A DEA	A LIV	-3.9000	0.59389	12	-6.57	0.0001
A DEA	B CON	1.2250	0.87281	5.98	1.40	0.2102
A DEA	B DEA	-1.4000	0.87281	5.98	-1.60	0.1600
A DEA	B LIV	-4.2750	0.87281	5.98	-4.90	0.0027
A LIV	B CON	5.1250	0.87281	5.98	5.87	0.0011
A LIV	B DEA	2.5000	0.87281	5.98	2.86	0.0288
A LIV	B LIV	-0.3750	0.87281	5.98	-0.43	0.6825
B CON	B DEA	-2.6250	0.59389	12	-4.42	0.0008
B CON	B LIV	-5.5000	0.59389	12	-9.26	0.0001
B DEA	B LIV	-2.8750	0.59389	12	-4.84	0.0004

General Linear Models Procedure

Dependent Variable: YIELD

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	11	157.208	14.292	20.26	0.0001
Error	12	8.465	0.705		
Corrected Total	23	165.673			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
CULTIVAR	1	2.40667	2.40667	3.41	0.0895
BLOCK	3	25.32000	8.44000	11.96	0.0006
BLOCK*CULTIVAR	3	9.48000	3.16000	4.48	0.0249
INNOC	2	118.17583	59.08792	83.76	0.0001
CULTIVAR*INNOC	2	1.82583	0.91292	1.29	0.3098

Source	DF	Type III SS	Mean Square	F Value	Pr > F
CULTIVAR	1	2.40667	2.40667	3.41	0.0895
BLOCK	3	25.32000	8.44000	11.96	0.0006
BLOCK*CULTIVAR	3	9.48000	3.16000	4.48	0.0249
INNOC	2	118.17583	59.08792	83.76	0.0001
CULTIVAR*INNOC	2	1.82583	0.91292	1.29	0.3098

```
Source            Type III Expected Mean Square

CULTIVAR          Var(Error) + 3 Var(BLOCK*CULTIVAR)
                       + Q(CULTIVAR,CULTIVAR*INNOC)

BLOCK             Var(Error) + 3 Var(BLOCK*CULTIVAR)
                       + 6 Var(BLOCK)

BLOCK*CULTIVAR    Var(Error) + 3 Var(BLOCK*CULTIVAR)

INNOC             Var(Error) + Q(INNOC,CULTIVAR*INNOC)

CULTIVAR*INNOC    Var(Error) + Q(CULTIVAR*INNOC)
```

Tests of Hypotheses using the Type III MS for BLOCK*CULTIVAR as an error term

Source	DF	Type III SS	Mean Square	F Value	Pr > F
CULTIVAR	1	2.40667	2.40667	0.76	0.4471
BLOCK	3	25.32000	8.44000	2.67	0.2206

Standard errors for sample means

Whole plot factor:

$Var(\bar{Y}_{i..})$

$$= Var\left(\frac{\sum_j \sum_k(\mu+\alpha_i+\beta_j+\eta_{ij}+\gamma_k+\delta_{ik}+e_{ijk})}{rb}\right)$$

$$= Var\left(\frac{1}{r}\sum_{j=1}^{r}\beta_j\right) + Var\left(\frac{1}{r}\sum_{j=1}^{r}\eta_{ij}\right) + Var\left(\frac{1}{br}\sum_{j=1}^{r}\sum_{k=1}^{b}e_{ijk}\right)$$

$$= \frac{\sigma_\beta^2}{r} + \frac{\sigma_w^2}{r} + \frac{\sigma_e^2}{rb}$$

$$= \frac{\sigma_e^2 + b\sigma_w^2 + b\sigma_\beta^2}{rb}$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$\hat{\sigma}_w^2 = \frac{MS_{block\times Cult} - MS_{error}}{3} = .8182$$

$$\hat{\sigma}_\beta^2 = \frac{MS_{blocks} - MS_{block\times cult}}{6} = .8800$$

$$S^2_{\bar{Y}_{i..}} = \frac{\hat{\sigma}_e^2 + b\hat{\sigma}_w^2 + b\hat{\sigma}_\beta^2}{rb}$$

$$= \frac{1}{rb}\left[\left(\frac{a-1}{a}\right)MS_{block*Cult} + \frac{1}{a}MS_{blocks}\right]$$

$$= 0.48333 \quad (\text{and } S_{\bar{Y}_{i..}} = 0.6952)$$

with

$$d.f. = \frac{\left[\frac{a-1}{a}MS_{block*cult} + \frac{1}{a}MS_{blocks}\right]^2}{\frac{\left[\frac{a-1}{a}MS_{blocks*clut}\right]^2}{(a-1)(r-1)} + \frac{\left[\frac{1}{a}MS_{blocks}\right]^2}{r-1}}$$

$$= 4.97$$

Difference in levels of the whole plot factor

$Var(\bar{Y}_{i..} - \bar{Y}_{s..})$

$$= Var\left(\frac{\sum_j\sum_k(Y_{ijk})}{rb} - \frac{\sum_j\sum_k(Y_{sjk})}{rb}\right)$$

$$= Var\left(\frac{1}{rb}\sum_j\sum_k(Y_{ijk}) - Y_{sjk})\right)$$

$$= Var\left(\frac{1}{rb}\sum_j\sum_k([\mu+\alpha_i+\beta_j+\eta_{ij}+\gamma_k+\delta_{ik}+e_{ijk}] - [\mu+\alpha_s+\beta_j+\eta_{sj}+\gamma_k+\delta_{sk}+e_{sjk}])\right)$$

$$= Var\left(\alpha_i - \alpha_s + \frac{1}{r}\sum_j(\eta_{ij}-\eta_{sj}) + \frac{1}{b}\sum_k(\delta_{ik}-\delta_{sk}) + \frac{1}{rb}\sum_j\sum_k(e_{ijk}-e_{sjk})\right)$$

$$= Var\left(\frac{1}{r}\sum_j(\eta_{ij}-\eta_{sj})\right) + Var\left(\frac{1}{rb}\sum_j\sum_k(e_{ijk}-e_{sjk})\right)$$

$$= \frac{2\sigma_w^2}{r} + \frac{2\sigma_e^2}{rb}$$

$$= \frac{\sigma_e^2 + b\sigma_w^2}{rb}$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$\hat{\sigma}_w^2 = \frac{1}{3}(MS_{block\times Cult} - MS_{error}) = .8182$$

$$S^2_{\bar{Y}_{i..}-\bar{Y}_{s..}} = \frac{2}{rb}(\hat{\sigma}_e^2 + b\hat{\sigma}_w^2)$$

$$= \frac{2}{rb} MS_{block*Cult}$$

$$= 0.526 \quad (\text{and } S_{\bar{Y}_{i..}-\bar{Y}_{s..}} = 0.726)$$

with

$$d.f. = 3$$

Subplot factor

$$Var(\bar{Y}_{..k})$$

$$= Var\left(\frac{\sum_i \sum_j (\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk})}{ar}\right)$$

$$= Var\left(\frac{1}{r}\sum_{j=1}^{r}\beta_j\right) + Var\left(\frac{1}{ar}\sum_{i=1}^{a}\sum_{j=1}^{r}\eta_{ij}\right) + Var\left(\frac{1}{ar}\sum_{1=1}^{a}\sum_{j=1}^{r}e_{ijk}\right)$$

$$= \frac{\sigma_\beta^2}{r} + \frac{\sigma_w^2}{ar} + \frac{\sigma_e^2}{ar}$$

$$= \frac{\sigma_e^2 + \sigma_w^2 + a\sigma_\beta^2}{ar}$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$\hat{\sigma}_w^2 = \frac{MS_{block\times Cult} - MS_{error}}{3} = .8182$$

$$\hat{\sigma}_\beta^2 = \frac{MS_{blocks} - MS_{block\times cult}}{6} = .8800$$

$$S^2_{\bar{Y}_{i..}} = \frac{\hat{\sigma}_e^2 + \hat{\sigma}_w^2 + a\hat{\sigma}_\beta^2}{ar}$$

$$= \frac{1}{ar}\left[\left(\frac{b-1}{b}\right) MS_{error} + \frac{1}{b} MS_{blocks}\right]$$

$$= .4104 \quad (\text{and } S_{\bar{Y}_{..k}} = 0.6407)$$

with

$$d.f. = \frac{\left[\frac{b-1}{b} MS_{error} + \frac{1}{b} MS_{blocks}\right]^2}{\frac{\left[\frac{b-1}{b} MS_{error}\right]^2}{12} + \frac{\left[\frac{1}{b} MS_{blocks}\right]^2}{3}}$$

$$= 4.06$$

Difference in levels of the subplot factor

$$Var(\bar{Y}_{..k} - \bar{Y}_{..\ell})$$

$$= Var\left(\frac{\sum_i \sum_j (Y_{ijk})}{ar} - \frac{\sum_i \sum_j (Y_{ij\ell})}{ar}\right)$$

$$= Var\left(\frac{1}{ar}\sum_i \sum_j (Y_{ijk} - Y_{ij\ell})\right)$$

$$= Var\left(\frac{1}{ar}\sum_i \sum_j ([\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}] - [\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_\ell + \delta_{i\ell} + e_{ij\ell}])\right)$$

$$= Var\left(\frac{1}{a}\sum_i (\delta_{ik} - \delta_{i\ell}) + (\gamma_k - \gamma_\ell) + \frac{1}{ar}\sum_i \sum_j (e_{ijk} - e_{ij\ell})\right)$$

$$= Var\left(\frac{1}{ar}\sum_j \sum_k (e_{ijk} - e_{ij\ell})\right)$$

$$= \frac{2\sigma_e^2}{ar}$$

Estimation:

$$\begin{aligned}
\hat{\sigma}_e^2 &= MS_{error} = .7054 \\
S^2_{\bar{Y}_{..k}-\bar{Y}_{..\ell}} &= \frac{2}{ar}(\hat{\sigma}_e^2) \\
&= \frac{2}{ar} MS_{error} \\
&= 0.1763 \quad (\text{and } S_{\bar{Y}_{..k}-\bar{Y}_{..\ell}} = 0.4199)
\end{aligned}$$

with

$$d.f. = 12$$

Difference in levels of the subplot factor for a specific level of the whole plot factor

$$\begin{aligned}
& Var(\bar{Y}_{i.k} - \bar{Y}_{i.\ell}) \\
&= Var\left(\frac{\sum_j (Y_{ijk})}{r} - \frac{\sum_j (Y_{ij\ell})}{r}\right) \\
&= Var\left(\frac{1}{r}\sum_j (Y_{ijk}) - Y_{ij\ell})\right) \\
&= Var\left(\frac{1}{r}\sum_j ([\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}]\right. \\
&\quad \left. -[\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_\ell + \delta_{i\ell} + e_{ij\ell}])\right) \\
&= Var\left(\gamma_k - \gamma_\ell + \delta_{ik} - \delta_{i\ell} \right. \\
&\qquad \left. + \frac{1}{r}\sum_j (e_{ijk} - e_{ij\ell})\right) \\
&= Var\left(\frac{1}{r}\sum_j (e_{ijk} - e_{ij\ell})\right) \\
&= \frac{2\sigma_e^2}{r}
\end{aligned}$$

Estimation:

$$\begin{aligned}
\hat{\sigma}_e^2 &= MS_{error} = .7054 \\
S^2_{\bar{Y}_{i.k}-\bar{Y}_{i.\ell}} &= \frac{2}{r}(\hat{\sigma}_e^2) \\
&= \frac{2}{r} MS_{error} \\
&= 0.3525 \quad (\text{and } S_{\bar{Y}_{i.k}-\bar{Y}_{i.\ell}} = 0.5937)
\end{aligned}$$

with

$$d.f. = 12$$

Difference in levels of the whole plot factor for a specific level of the subplot factor

$$\begin{aligned}
& Var(\bar{Y}_{i.k} - \bar{Y}_{s.k}) \\
&= Var\left(\frac{\sum_j (Y_{ijk})}{r} - \frac{\sum_j (Y_{sjk})}{r}\right) \\
&= Var\left(\frac{1}{r}\sum_j (Y_{ijk}) - Y_{sjk})\right) \\
&= Var\left(\frac{1}{r}\sum_j ([\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}]\right. \\
&\quad \left. -[\mu + \alpha_s + \beta_j + \eta_{sj} + \gamma_k + \delta_{sk} + e_{sjk}])\right) \\
&= Var\left(\alpha_i - \alpha_s + \delta_{ik} - \delta_{sk} \right. \\
&\qquad \left. + \frac{1}{r}\sum_j (\eta_{ij} - \eta_{sj}) + \frac{1}{r}\sum_j (e_{ijk} - e_{sjk})\right) \\
&= Var\left(\frac{1}{r}\sum_j (\eta_{ij} - \eta_{sj})\right) + Var\left(\frac{1}{r}\sum_j (e_{ijk} - e_{sjk})\right) \\
&= \frac{2}{r}(\sigma_w^2 + \sigma_e^2)
\end{aligned}$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$\hat{\sigma}_w^2 = \frac{MS_{block\times Cult} - MS_{error}}{3} = .8182$$

$$S^2_{\bar{Y}_{i.k}-\bar{Y}_{s.k}} = \frac{2}{r}(\hat{\sigma}_w^2 + \hat{\sigma}_e^2)$$

$$= \frac{2}{r}\left(\frac{b-1}{b}MS_{error} + \frac{1}{b}MS_{block\times Cult}\right)$$

$$= 0.7618 \quad \text{(and } S_{\bar{Y}_{i.k}-\bar{Y}_{s.k}} = 0.8728\text{)}$$

with

$$d.f. = \frac{\left[\frac{b-1}{b}MS_{error} + \frac{1}{b}MS_{block\times Cult}\right]^2}{\frac{\left[\frac{b-1}{b}MS_{error}\right]^2}{12} + \frac{\left[\frac{1}{b}MS_{block\times Cult}\right]^2}{3}}$$

$$= 5.98$$

Difference in levels of the subplot factor for different levels of the whole plot factor

$$Var(\bar{Y}_{i.k} - \bar{Y}_{s.\ell})$$

$$= Var\left(\frac{\sum_j(Y_{ijk})}{r} - \frac{\sum_j(Y_{sj\ell})}{r}\right)$$

$$= Var\left(\frac{1}{r}\sum_j(Y_{ijk}) - Y_{sj\ell})\right)$$

$$= Var\left(\frac{1}{r}\sum_j([\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}] - [\mu + \alpha_s + \beta_j + \eta_{sj} + \gamma_\ell + \delta_{s\ell} + e_{sj\ell}])\right)$$

$$= Var\left(\alpha_i - \alpha_s + \gamma_k - \gamma_\ell + \delta_{ik} - \delta_{s\ell} + \frac{1}{r}\sum_j(\eta_{ij} - \eta_{sj}) + \frac{1}{r}\sum_j(e_{ijk} - e_{sj\ell})\right)$$

$$= Var\left(\frac{1}{r}\sum_j(\eta_{ij} - \eta_{sj})\right) + Var\left(\frac{1}{r}\sum_j(e_{ijk} - e_{sj\ell})\right)$$

$$= \frac{2}{r}(\sigma_w^2 + \sigma_e^2)$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$\hat{\sigma}_w^2 = \frac{MS_{block\times Cult} - MS_{error}}{3} = .8182$$

$$S^2_{\bar{Y}_{i.k}-\bar{Y}_{s.\ell}} = \frac{2}{r}(\hat{\sigma}_w^2 + \hat{\sigma}_e^2)$$

$$= \frac{2}{r}\left(\frac{b-1}{b}MS_{error} + \frac{1}{b}MS_{block\times Cult}\right)$$

$$= 0.7618 \quad \text{(and } S_{\bar{Y}_{i.k}-\bar{Y}_{s.\ell}} = 0.8728\text{)}$$

with

$$d.f. = \frac{\left[\frac{b-1}{b}MS_{error} + \frac{1}{b}MS_{block\times Cult}\right]^2}{\frac{\left[\frac{b-1}{b}MS_{error}\right]^2}{12} + \frac{\left[\frac{1}{b}MS_{block\times Cult}\right]^2}{3}}$$

$$= 5.98$$

Interaction Contrasts

$$Var(\bar{Y}_{i.k} - \bar{Y}_{i.\ell} - \bar{Y}_{s.k} + \bar{Y}_{s.\ell})$$

$$= Var\left(\frac{1}{r}\sum_j(Y_{ijk} - Y_{ij\ell} - Y_{sjk} + Y_{sj\ell})\right)$$

$$= Var\left(\frac{1}{r}\sum_j([\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_k + \delta_{ik} + e_{ijk}] - [\mu + \alpha_i + \beta_j + \eta_{ij} + \gamma_\ell + \delta_{i\ell} + e_{ij\ell}] - [\mu + \alpha_s + \beta_j + \eta_{sj} + \gamma_k + \delta_{sk} + e_{sjk}] + [\mu + \alpha_s + \beta_j + \eta_{sj} + \gamma_\ell + \delta_{s\ell} + e_{sj\ell}])\right)$$

$$= Var\left(\delta_{ik} - \delta_{i\ell} - \delta_{sk} + \delta_{s\ell} + \frac{1}{r}\sum_j(e_{ijk} - e_{ij\ell} - e_{sjk} + e_{sj\ell})\right)$$

$$= \frac{4\sigma_e^2}{r}$$

Estimation:

$$\hat{\sigma}_e^2 = MS_{error} = .7054$$

$$S^2_{\bar{Y}_{i.k}-\bar{Y}_{i.\ell}-\bar{Y}_{s.k}+\bar{Y}_{s.\ell}} = \frac{4}{r}(\hat{\sigma}_e^2) = \frac{4}{r}MS_{error} = 0.705$$

and

$$S_{\bar{Y}_{i.k}-\bar{Y}_{i.\ell}-\bar{Y}_{s.k}+\bar{Y}_{s.\ell}} = 0.5396)$$

with

$$d.f. = 12$$

Example 10.4: Repeated Measures

In an exercise therapy study, subjects were assigned to one of three weightlifting programs

(i=1) The number of repetitions of weightlifting was increased as subjects became stronger (RI)

(i=2) The amount of weight was increased as subjects became stronger (WI)

(i=3) Subjects did not participate in weightlifting (XCont)

Measurements of strength (Y) were taken on days 2, 4, 6, 8, 10, 12 and 14 for each subject.

Source:
Littel, Freund, and Spector (1991) SAS System for Linear Models

Data: weight2.dat

SAS code: weight2.sas

S-PLUS code: weight2.ssc

Mixed model

$$Y_{ijk} = \mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}$$

- Y_{ijk} strength measurement at the k-th time point for the j-th subject in the i-th program
- α_i "fixed" program effect
- S_{ij} random subject effect
- τ_k "fixed" time effect
- e_{ijk} random error

where the random effects are all independent and

$$S_{ij} \sim NID(0, \sigma_S^2)$$
$$\epsilon_{ijk} \sim NID(0, \sigma_\epsilon^2)$$

Average strength after $2k$ days on the i-th program is

$$\begin{aligned} \mu_{ik} &= E(Y_{ijk}) \\ &= E(\mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}) \\ &= \mu + \alpha_i + E(S_{ij}) + \tau_k + \gamma_{ik} + E(e_{ijk}) \\ &= \mu + \alpha_i + \tau_k + \gamma_{ik} \end{aligned}$$

for $i = 1, 2, 3$ and $k = 1, 2, \ldots, 7$

The variance of any single observation is

$$\begin{aligned} Var(Y_{ijk}) &= Var(\mu + \alpha_i + S_{ij} + \tau_k + \alpha_{ik} + e_{ijk}) \\ &= Var(S_{ij} + e_{ijk}) \\ &= Var(S_{ij}) + Var(e_{ijk}) \\ &= \sigma_S^2 + \sigma_e^2 \end{aligned}$$

Correlation between observations taken on the same subject:

$$\begin{aligned} & Cov(Y_{ijk}, Y_{ij\ell}) \\ &= Cov(\mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}, \\ &\qquad \mu + \alpha_i + S_{ij} + \tau_\ell + \gamma_{i\ell} + e_{ij\ell}) \\ &= Cov(S_{ij} + e_{ijk}, S_{ij} + e_{ij\ell}) \\ &= Cov(S_{ij}, S_{ij}) + Cov(S_{ij}, e_{ij\ell}) \\ &\quad + Cov(e_{ijk}, S_{ij}) + Cov(e_{ijk}, e_{ij\ell}) \\ &= Var(S_{ij}) \\ &= \sigma_S^2 \quad \text{for } k \neq \ell. \end{aligned}$$

The correlation between Y_{ijk} and $Y_{ij\ell}$ is

$$\frac{\sigma_S^2}{\sigma_S^2 + \sigma_e^2} \equiv \rho$$

Observations taken on different subjects are uncorrelated.

For the set of observations taken on a single subject, we have

$$Var\left(\begin{bmatrix} Y_{ij1} \\ Y_{ij3} \\ \vdots \\ Y_{ij7} \end{bmatrix}\right) = \begin{bmatrix} \sigma_e^2 + \sigma_S^2 & \sigma_S^2 & \cdots & \sigma_S^2 \\ \sigma_S^2 & \sigma_e^2 + \sigma_S^2 & \cdots & \sigma_S^2 \\ \vdots & \vdots & \ddots & \sigma_S^2 \\ \sigma_S^2 & \sigma_S^2 & \sigma_S^2 & \sigma_e^2 + \sigma_S^2 \end{bmatrix}$$

$$= \sigma_e^2 I + \sigma_S^2 J$$

$\uparrow$

J is a matrix of ones

This covariance structure is called compound symmetry.

Write this model in the form

$$\mathbf{Y} = X\beta + Z\mathbf{u} + \mathbf{e}$$

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ \vdots \\ Y_{117} \\ Y_{121} \\ Y_{122} \\ \vdots \\ Y_{127} \\ \vdots \\ Y_{211} \\ Y_{212} \\ \vdots \\ Y_{217} \\ \vdots \\ Y_{3,n_3 1} \\ Y_{3,n_3 2} \\ \vdots \\ Y_{3,n_3 7} \end{bmatrix} = \left[\begin{array}{cccc|c|c} 1 & 1 & 0 & 0 & 1000000 & \\ 1 & 1 & 0 & 0 & 0100000 & \\ & \vdots & & & \ddots & \text{21} \\ 1 & 1 & 0 & 0 & 0000001 & \text{columns} \\ 1 & 1 & 0 & 0 & 1000000 & \text{for} \\ 1 & 1 & 0 & 0 & 0100000 & \text{interaction} \\ & \vdots & & & \ddots & \\ 1 & 1 & 0 & 0 & 0000001 & \\ \vdots & & & & & \\ 1 & 0 & 1 & 0 & 1000000 & \\ 1 & 0 & 1 & 0 & 0100000 & \\ & & \vdots & & \ddots & \\ 1 & 0 & 1 & 0 & 0000001 & \\ \vdots & & & & & \\ 1 & 0 & 0 & 1 & 1000000 & \\ 1 & 0 & 0 & 1 & 0100000 & \\ \vdots & & \vdots & & \ddots & \\ 1 & 0 & 0 & 1 & 0000001 & \end{array}\right] \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \\ \tau_7 \\ \gamma_{11} \\ \gamma_{12} \\ \vdots \\ \gamma_{37} \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} S_{11} \\ S_{12} \\ \vdots \\ \vdots \\ \vdots \\ S_{3,n_3} \end{bmatrix} + \begin{bmatrix} e_{111} \\ e_{112} \\ \vdots \\ e_{117} \\ e_{121} \\ e_{122} \\ \vdots \\ e_{127} \\ \vdots \\ e_{211} \\ e_{212} \\ \vdots \\ e_{217} \\ \vdots \\ e_{3,n_3 1} \\ e_{3,n_3 2} \\ \vdots \\ e_{3,n_3 7} \end{bmatrix}$$

In this case:

$$R = Var(\mathbf{e}) \;=\; \sigma_e^2 I_{(7r)\times(7r)}$$

$$G = Var(\mathbf{u}) \;=\; \sigma_S^2 I_{r\times r}$$

where r is the number of subjects

$$\Sigma = Var(\mathbf{Y}) \;=\; ZGZ^T + R$$

is a block diagonal matrix with one block of the form $(\sigma_e^2 I_{7\times 7} + \sigma_S^2 J_{7\times 7})$ for each subject


```
/*  SAS code for analyzing repeated measures data
    across time(longitudinal studies).  This code
    is applied to the weightlifting data from
    Littel, et. al. (1991)  This code is posted
    on the course web page under

                  weight2.sas     */


data set1;
   infile 'weight2.dat';
   input subj program$ s1 s2 s3 s4 s5 s6 s7;
   if program='XCONT' then cprogram=3;
   if program='RI' then cprogram=1;
   if program='WI' then cprogram=2;
   run;;


/* Create a data file where responses at different
   time points are on different lines  */

data set2;
   set set1;
   time=2; strength=s1; output;
   time=4; strength=s2; output;
   time=6; strength=s3; output;
   time=8; strength=s4; output;
   time=10; strength=s5; output;
   time=12; strength=s6; output;
   time=14; strength=s7; output;
   keep subj program cprogram time strength;
run;
```

```
/*Create a profile plot with time on the horizontal axis*/

proc sort data=set2; by cprogram time;
run;

proc means data=set2 noprint;
   by cprogram time;
   var strength;
   output out=seta mean=strength;
run;

goptions  device=WIN target=WINPRTC rotate=landscape;

/* UNIX users can replace the previous line
   with the following lines  */

/* filename graffile pipe 'lpr -Dpostscript';
   goptions gsfmode=replace gsfname=graffile
   targetdevice=ps300 rotate=landscape; */

   axis1 label=(f=swiss h=2.5)
      value=(f=swiss h=2.0)  w=3.0  length= 5.5 in;

   axis2 label=(f=swiss h=2.2 a=270 r=90)
      value=(f=swiss h=2.0) w= 3.0  length = 4.2 in;

       SYMBOL1 V=circle H=2.0 w=3 l=1 i=join ;
       SYMBOL2 V=diamond H=2.0 w=3 l=3 i=join ;
       SYMBOL3 V=square H=2.0 w=3 l=9 i=join ;
```



```
 proc gplot data=seta;
   plot strength*time=cprogram / vaxis=axis2 haxis=axis1;
   title  H=3.0 F=swiss "Observed Strength Means";
       label strength=' ';
       label time = 'Time (days) ';
 run;

/*  Fit the standard mixed model with a
    compound symmetric covariance structure
    by specifying a random subject effect  */

proc mixed data=set2;
   class program subj time;
   model strength = program time program*time /
         dfm=satterth;
   random subj(program) / cl;
   lsmeans program / pdiff tdiff;
   lsmeans time / pdiff tdiff;
   lsmeans program*time / slice=time pdiff tdiff;
run;


/* Use the GLM procedure in SAS to get formulas for
   expectations of mean squares.  */


proc glm data=set2;
  class program subj time;
  model strength = program subj(program) time program*time;
  random subj(program);
  lsmeans program / pdiff tdiff;
  lsmeans time / pdiff tdiff;
  lsmeans program*time / slice=time pdiff tdiff;
run;
```

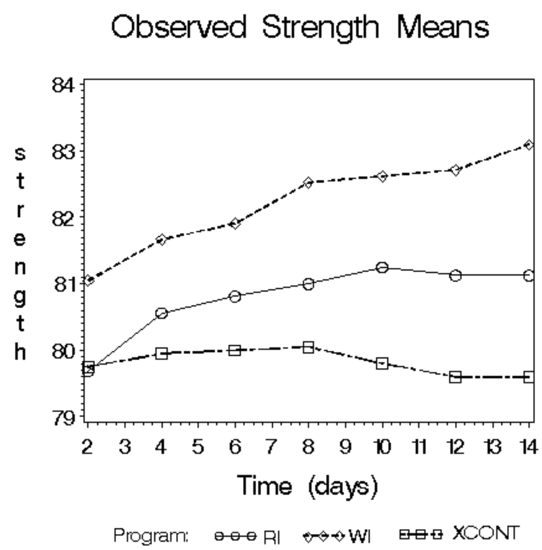


```
/* Fit the same model with the repeated statement.
   Here the compound symmetry covariance structure
   is selected.                                  */

proc mixed data=set2;
   class program subj time;
   model strength = program time program*time;
   repeated / type=cs sub=subj(program) r rcorr;
run;


/* Fit a model with the same fixed effects,
   but change the covariance structure to
   an AR(1) model                          */

proc mixed data=set2;
   class program subj time;
   model strength = program time program*time;
   repeated / type=ar(1) sub=subj(program) r rcorr;
run;
```



```
/* Fit a model with the same fixed effects,
   but use an arbitary covariance matrix
   for repeated measures on the same subject. */

proc mixed data=set2;
   class program subj time;
   model strength = program time program*time;
   repeated / type=un sub=subj(program) r rcorr;
run;


/*  Fit a model with linear and quadratic
    trends across time and different trends
    across time for different programs.   */

proc mixed data=set2;
   class program subj;
   model strength = program time time*program
   time*time time*time*program / htype=1;
   repeated / type=ar(1) sub=subj(program);
run;

/*  Fit a model with linear, quadratic and
    cubic trends across time and different
    trends across time for different programs. */

proc mixed data=set2;
   class program subj;
   model strength = program time time*program
   time*time time*time*program
   time*time*time time*time*time*program / htype=1;
   repeated / type=ar(1) sub=subj(program);
run;
```

```
/* By removing the automatic intercept and adding
   the solution option to the model statement, the
   estimates of the parameters in the model are obtained.
   Here we use a power function model for the covariance
   structure.  This is a generalization of the AR(1)
   covariance structure that can be used with unequally
   spaced time points. */

proc mixed data=set2;
   class program subj;
   model strength = program time*program time*time*program
                     / noint solution htype=1;
   repeated / type=sp(pow)(time) sub=subj(program) r rcorr
run;


/*  Fit a model with random coefficients that allow
    individual subjects to have different linear and
    quadratic trends across time.  */


proc mixed data=set2 scoring=8;
   class program subj;
   model strength = program time*program time*time*program
                       noint solution htype=1;
   random int time time*time / type=un sub=subj(program)
                            solution ;
run;
```


```
                     The Mixed Procedure

                   Class Level Information

     Class      Levels    Values

     program         3    RI WI XCONT
     subj           21    1 2 3 4 5 6 7 8 9 10 11 12 13
                          14 15 16 17 18 19 20 21
     time            7    2 4 6 8 10 12 14


                      Iteration History

 Iteration    Evaluations    -2 Res Log Like   Criterion

         0              1      2033.882983
         1              1      1420.820196       0.00000

                  Convergence criteria met.

                Covariance Parameter Estimates

                   Cov Parm          Estimate

                   subj(program)       9.6033
                   Residual            1.1969
```



```
                   Fitting Information

     Res Log Likelihood                       -710.4
     Akaike's Information Criterion           -712.4
     Schwarz's Bayesian Criterion             -714.5
     -2 Res Log Likelihood                    1420.8


               Solution for Random Effects


Effect             Estimate   Pred   DF       t    Pr>|t|

subj(prog)  RI  1  -1.4825  0.8705  81.6  -1.70  0.0923
subj(prog)  RI  2   4.2722  0.8705  81.6   4.91  <.0001
subj(prog)  RI  3   1.4650  0.8705  81.6   1.68  0.0962
     .
     .
     .
subj(prog)  XC 20  -0.2456  0.7998  90.1  -0.31  0.7595
```

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	2	54	3.07	0.0548
time	6	324	7.43	<.0001
program*time	12	324	2.99	0.0005

Least Squares Means

Effect	Estimate	Standard Error	DF	t Value	Pr>|t|
program RI	80.7946	0.7816	54	103.37	<.0001
program WI	82.2245	0.6822	54	120.52	<.0001
program XCONT	79.8214	0.6991	54	114.18	<.0001
time 2	80.1617	0.4383	65.8	182.87	<.0001
time 4	80.7264	0.4383	65.8	184.16	<.0001
time 6	80.9058	0.4383	65.8	184.57	<.0001
time 8	81.1913	0.4383	65.8	185.22	<.0001
time 10	81.2230	0.4383	65.8	185.29	<.0001
time 12	81.1464	0.4383	65.8	185.12	<.0001
time 14	81.2734	0.4383	65.8	185.41	<.0001
prog*time RI 2	79.6875	0.8216	65.8	96.99	<.0001
prog*time RI 4	80.5625	0.8216	65.8	98.06	<.0001
.				.	.
.				.	.
.				.	.
prog*time XC 14	79.6000	0.7349	65.8	108.32	<.0001

Differences of Least Squares Means

Level1	Level2	Estimate	Standard Error	DF	t Value	Pr>|t|
RI	WI	-1.4298	1.0375	54	-1.38	0.1738
RI	XCONT	0.9732	1.0486	54	0.93	0.3575
WI	XCONT	2.4031	0.9768	54	2.46	0.0171
time 2	4	-0.5647	0.2064	324	-2.74	0.0066
time 2	6	-0.7440	0.2064	324	-3.61	0.0004
.						
.						
.						
time 12	14	-0.1270	0.2064	324	-0.62	0.5388
RI 2	RI 4	-0.8750	0.3868	324	-2.26	0.0243
RI 2	RI 6	-1.1250	0.3868	324	-2.91	0.0039
.						
.						
.						
RI 4	XCONT 2	0.8125	1.1023	65.8	0.74	0.4637
.						
.						
.						
XC 12	XCONT 14	1.24E-13	0.3460	324	0.00	1.0000

Mean strength at a particular time in a particular program

$$LSMEAN = \hat{\mu} + \hat{\alpha}_i + \hat{\tau}_k + \hat{\gamma}_{ik}$$
$$= \bar{Y}_{i.k} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ijk}$$

$$Var(\bar{Y}_{i.k}) = \frac{\sigma_e^2 + \sigma_S^2}{n_i}$$

$$S_{\bar{Y}_{i.k}} = \sqrt{\left(\frac{6}{7}MS_{error} + \frac{1}{7}MS_{Subj}\right)\frac{1}{n_i}}$$

↑ Cochran-Satterthwaite degrees of freedom are 65.8

Program means (averaging across time)

$$LSMEAN = \bar{Y}_{i..}$$
$$= \hat{\mu} + \hat{\alpha}_i + \frac{1}{7}\sum_{k=1}^{7}(\hat{\tau}_k + \hat{\gamma}_{ik})$$

$$Var(\bar{Y}_{i...}) = \frac{\sigma_e^2 + 7\sigma_S^2}{7n_i}$$

$$S_{\bar{Y}_{i..}} = \sqrt{\frac{MS_{subjects}}{2n_i}}$$

There are $n_1 = 16,\ n_2 = 21,\ n_3 = 20$ subjects in the three programs.

Mean strength at a particular time point (averaging across programs)

$$LSMEAN = \hat{\mu} + \hat{\tau}_k + \frac{1}{3}\sum_{i=1}^{3}(\hat{\alpha}_i + \hat{\gamma}_{ik}) = \frac{1}{3}\sum_{i=1}^{3}(\bar{Y}_{i.k}) \neq \bar{Y}_{..k}$$

because $n_1 = 16,\ n_2 = 21,\ n_3 = 20.$

$$Var(LSMEAN) = \frac{1}{9}\sum_{i=1}^{3}\frac{\sigma_e^2 + \sigma_S^2}{n_i}$$

$$S_{LSMEAN} = \frac{1}{3}\sqrt{\left(\frac{6}{7}MS_{error} + \frac{1}{7}MS_{subj}\right)\left(\sum_{i=1}^{3}\frac{1}{n_i}\right)}$$

↑
Cochran-Satterthwaite
degrees of freedom are 65.8

Difference between strength means at two time points (averaging across programs)

$$\left(\hat{\tau}_k + \frac{1}{3}\Sigma_{i=1}^{3}\hat{\gamma}_{ik}\right) - \left(\hat{\tau}_\ell - \frac{1}{3}\Sigma_{i=1}^{3}\hat{\gamma}_{i\ell}\right)$$

$$= \frac{1}{3}\left(\sum_{i=1}^{3}\bar{Y}_{i.k}\right) - \frac{1}{3}\left(\sum_{i=1}^{3}\bar{Y}_{i.\ell}\right)$$

$$= \frac{1}{3}\sum_{i=1}^{3}\frac{1}{n_i}\sum_{j=1}^{n_i}\underline{(Y_{ijk} - Y_{ij\ell})}$$

↑
subject effects
cancel out

Variance formula:

$$\frac{2\sigma_e^2}{9}\Sigma_{i=1}^{3}\frac{1}{n_i}$$

Standard error:

$$\sqrt{\frac{2MS_{error}}{9}\ \Sigma_{i=1}^{3}\frac{1}{n_i}} = 0.206$$

↑
use degrees of freedom
for error $= 324$

Difference between strength means for two programs at a specific time point

$$(\hat{\mu} + \hat{\alpha}_i + \hat{\tau}_{ik} + \hat{\gamma}_{ik}) - (\hat{\mu} + \hat{\alpha}_\ell + \hat{\tau}_k + \hat{\gamma}_{\ell k})$$

$$= \bar{Y}_{i.k} - \bar{Y}_{\ell.k}$$

$$Var(\bar{Y}_{i.k} - \bar{Y}_{\ell.k}) = (\sigma_e^2 + \sigma_S^2)\left(\frac{1}{n_i} + \frac{1}{n_\ell}\right)$$

$$S_{\bar{Y}_{i.k} - \bar{Y}_{\ell.k}} = \sqrt{\left(\frac{6}{7}MS_{error} + \frac{1}{7}MS_{subj}\right)\left(\frac{1}{n_i} + \frac{1}{n_\ell}\right)}$$

↑
Use Cochran-Satterthwaite
degrees of freedom $= 65.8$

Difference between strength means at two time points within a particular program

$$(\hat{\mu} + \hat{\alpha}_i + \hat{\tau}_k + \hat{\gamma}_{ik}) - (\hat{\mu} + \hat{\alpha}_i + \hat{\tau}_\ell + \hat{\gamma}_{i\ell})$$

$$= \bar{Y}_{i.k} - \bar{Y}_{i.\ell}$$

$$Var(\bar{Y}_{i.k} - \bar{Y}_{i.\ell}) = Var\left(\frac{1}{n_i}\Sigma_{j=1}^{n_i}(Y_{ijk} - Y_{ij\ell})\right) = \frac{2\sigma_e^2}{n_i}$$

$$S_{\bar{Y}_{i.k}-\bar{Y}_{i.\ell}} = \sqrt{MS_{error}\left(\frac{2}{n_i}\right)}$$

↑ use degrees of freedom for error=324

Difference in strength means for two programs (averaging across time points)

$$\bar{Y}_{i..} - \bar{Y}_{\ell..} = (\hat{\alpha}_i - \frac{1}{7}\sum_{k=1}^{7}\hat{\gamma}_{ik}) - (\hat{\alpha}_\ell + \frac{1}{7}\sum_{k=1}^{7}\hat{\gamma}_{\ell k})$$

$$Var(\bar{Y}_{i..} - \bar{Y}_{\ell..}) = (\sigma_e^2 + 7\sigma_S^2)(\frac{1}{7n_i} + \frac{1}{7n_\ell})$$

$$S_{\bar{Y}_{i..}-\bar{Y}_{\ell..}} = \sqrt{MS_{subjects}(\frac{1}{7n_i} + \frac{1}{7n_\ell})}$$

↑ 54 d.f.

Tests of Effect Slices

Effect	time	Num DF	Den DF	F Value	Pr > F
program*time	2	2	65.8	1.08	0.3455
program*time	4	2	65.8	1.44	0.2454
program*time	6	2	65.8	1.73	0.1844
program*time	8	2	65.8	2.96	0.0590
program*time	10	2	65.8	3.77	0.0282
program*time	12	2	65.8	4.60	0.0135
program*time	14	2	65.8	5.83	0.0047

The GLM Procedure

Class Level Information

Class	Levels	Values
program	3	RI WI XCONT
subj	21	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
time	7	2 4 6 8 10 12 14

Number of observations 399

Dependent Variable: strength

Source	DF	Sum of Squares	Mean Square	F	Pr > F
Model	74	4210.0529	56.8926	47.53	<.0001
Error	324	387.7867	1.1969		
C Total	398	4597.8396			

Source	DF	Type III SS	Mean Square	F	Pr > F
program	2	419.4352	209.7176	175.22	<.0001
subj(program)	54	3694.6900	68.4202	57.17	<.0001
time	6	52.9273	8.8212	7.37	<.0001
program*time	12	43.0002	3.5834	2.99	0.0005

Source	Type III Expected Mean Square
program	Var(Error) + 7 Var(subj(program)) + Q(program,program*time)
subj(program)	Var(Error) + 7 Var(subj(program))
time	Var(Error) + Q(time,program*time)
program*time	Var(Error) + Q(program*time)

Specifying other covariance matrices

We began with the model

$$Y_{ijk} = \mu + \alpha_i + S_{ij} + \tau_k + \gamma_{ik} + e_{ijk}$$

where

$$S_{ij} \sim NID(0, \sigma_S^2)$$
$$e_{ijk} \sim NID(0, \sigma_e^2)$$

and the $\{S_{ij}\}$ are distributed independently of the $\{e_{ijk}\}$

This model was expressed in the form

$$\mathrm{Y} = X\beta + Z\mathrm{u} + \mathrm{e}$$

where

$$\mathrm{u} = \begin{bmatrix} S_{11} \\ \vdots \\ S_{3,20} \end{bmatrix}$$

contained the random subject effects

Here

$$G = Var(\mathrm{u}) \;=\; \sigma_S^2 I$$

$$R = Var(\mathrm{e}) \;=\; \sigma_e^2 I$$

$$\Sigma = Var(\mathrm{Y}) \;=\; ZGZ^T + R$$

$$= \sigma_S^2 \begin{bmatrix} J & & & \\ & J & & \\ & & \ddots & \\ & & & J \end{bmatrix} + \sigma_e^2 I$$

$$= \begin{bmatrix} \sigma_e^2 I + \sigma_S^2 J & & \\ & \ddots & \\ & & \sigma_e^2 I + \sigma_S^2 J \end{bmatrix}$$

where J is a matrix of ones.

If you are not interested in predicting subject effects (random subject effects are included only to introduce correlation among repeated measures on the same subject), you can work with an alternative expression of the same model

$$Y = X\beta + e^*$$

where

$$\begin{aligned} R &= Var(e^*) \\ &= \begin{bmatrix} \sigma_e^2 I + \sigma_S^2 J & & \\ & \ddots & \\ & & \sigma_e^2 I + \sigma_S^2 J \end{bmatrix} \end{aligned}$$

Replace the mixed model

$$Y = X\beta + Zu + e$$

with the model

$$Y = X\beta + e^*$$

where

$$Var(Y) = Var(e^*) = \begin{bmatrix} W & 0 & \cdots & 0 \\ 0 & W & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & W \end{bmatrix}$$

You can specify this model by using the REPEATED statement in PROC MIXED

REPEATED / type =
subject = subj(program)

↗ ↗
variables in the class statement

r rcorr;

↗ print the W matrix for one subject

↗ print the correlation martix for one subject

Compound Symmetry: (type = CS)

$$W = \begin{bmatrix} \sigma_1^2+\sigma_2^2 & \sigma_2^2 & \sigma_2^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_1^2+\sigma_2^2 & \sigma_2^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_2^2 & \sigma_1^2+\sigma_2^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_2^2 & \sigma_2^2 & \sigma_1^2+\sigma_2^2 \end{bmatrix}$$

Variance components: (type = VC) (default)

$$W = \begin{bmatrix} \sigma_1^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2 \end{bmatrix}$$

Unstructured: (type = UN)

$$W = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \sigma_{34} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_4^2 \end{bmatrix}$$

Toeplitz: (type = TOEP)

$$W = \begin{bmatrix} \sigma^2 & \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1 & \sigma^2 & \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 & \sigma^2 & \sigma_1 \\ \sigma_3 & \sigma_2 & \sigma_1 & \sigma^2 \end{bmatrix}$$

Heterogeneous Toeplitz:
(type = TOEPH)

$$W = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_1 & \sigma_1\sigma_2\rho_2 & \sigma_1\sigma_4\rho_3 \\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_1 & \sigma_2\sigma_4\rho_2 \\ \sigma_3\sigma_1\rho_2 & \sigma_3\sigma_2\rho_1 & \sigma_3^2 & \sigma_3\sigma_4\rho_1 \\ \sigma_4\sigma_1\rho_3 & \sigma_4\sigma_2\rho_2 & \sigma_4\sigma_3\rho_1 & \sigma_4^2 \end{bmatrix}$$

First order Ante-dependence:
(type = ANTE(1))

$$W = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho_1 & \sigma_1\sigma_3\rho_1\rho_2 \\ \sigma_2\sigma_1\rho_1 & \sigma_2^2 & \sigma_2\sigma_3\rho_2 \\ \sigma_3\sigma_1\rho_2\rho_1 & \sigma_3\sigma_2\rho_2 & \sigma_3^2 \end{bmatrix}$$

First Order Autoregressive:
(type = AR(1))

$$W = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}$$

Heterogeneous AR(1): (type = ARH(1))

$$W = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho & \sigma_1\sigma_3\rho^2 & \sigma_1\sigma_4\rho^3 \\ \sigma_2\sigma_1\rho & \sigma_2^2 & \sigma_2\sigma_3\rho & \sigma_2\sigma_4\rho^2 \\ \sigma_3\sigma_1\rho^2 & \sigma_3\sigma_2\rho & \sigma_3^2 & \sigma_3\sigma_4\rho \\ \sigma_4\sigma_1\rho^3 & \sigma_4\sigma_2\rho^2 & \sigma_4\sigma_3\rho & \sigma_4^2 \end{bmatrix}$$

Spatial power: (type = sp(pow)(list))

↗ **list of variables defining coordinates**

$$W = \sigma^2 \begin{bmatrix} 1 & \rho^{d_{12}} & \rho^{d_{13}} & \rho^{d_{14}} \\ \rho^{d_{12}} & 1 & \rho^{d_{23}} & \rho^{d_{24}} \\ \rho^{d_{13}} & \rho^{d_{23}} & 1 & \rho^{d_{34}} \\ \rho^{d_{14}} & \rho^{d_{24}} & \rho^{d_{34}} & 1 \end{bmatrix}$$

where d_{ij} is the Euclidean distance between the i-th and j-th observations provided by one subject (or unit).

You can replace *pow* with a number of other choices.

Selecting a Covariance Structure

Assume $E(\mathrm{Y}) = X\beta)$ is correct.

- Likelihood ratio tests for "nested" models

$$-2\begin{pmatrix}\text{REML log-likelihood for}\\ \text{the smaller model}\end{pmatrix}$$
$$-\left[-2\begin{pmatrix}\text{REML log-likelihood for}\\ \text{the larger model}\end{pmatrix}\right]$$

$$\dot\sim\chi^2_{df} \quad \text{for large } n$$

where

$$df = \begin{bmatrix}\text{number of covariance}\\ \text{parameters in the}\\ \text{larger model}\end{bmatrix} - \begin{bmatrix}\text{number of covariance}\\ \text{parameters in the}\\ \text{smaller model}\end{bmatrix}$$

Consider models with

- Larger values of the Akaike Information Criterion(AIC)

  (REML Log-likelihood)

$$-\begin{pmatrix}\text{number of parameters in}\\ \text{the covariance model}\end{pmatrix}$$

- Larger values of the Schwarz Bayesian Criterion (SBC)

  (REML log-likelihood)

$$-\frac{log(n-p)}{2}\begin{pmatrix}\text{number of parameters}\\ \text{in the covariance model}\end{pmatrix}$$

  Here $n = 7 \times 57 = 399$ observations
  $p = 21$ parameters in $X\beta$


```
              The Mixed Procedure

                Model Information

    Data Set                      WORK.SET2
    Dependent Variable            strength
    Covariance Structure          Compound Symmetry
    Subject Effect                subj(program)
    Estimation Method             REML
    Residual Variance Method      Profile
    Fixed Effects SE Method       Model-Based
    Degrees of Freedom Method     Between-Within


             Class Level Information

    Class      Levels    Values

    program         3    RI WI XCONT
    subj           21    1 2 3 4 5 6 7 8 9 10 11 12 13
                         14 15 16 17 18 19 20 21
    time            7    2 4 6 8 10 12 14


                        Iteration History

 Iteration    Evaluations    -2 Res Log Like     Criterion

        0              1     2033.88298356
        1              1     1420.82019617      0.00000000
```



```
      Estimated R Matrix for subj(program) 1 RI

Row    Col1     Col2     Col3     Col4     Col5     Col6     Col7

  1 10.8002   9.6033   9.6033   9.6033   9.6033   9.6033   9.6033
  2  9.6033  10.8002   9.6033   9.6033   9.6033   9.6033   9.6033
  3  9.6033   9.6033  10.8002   9.6033   9.6033   9.6033   9.6033
  4  9.6033   9.6033   9.6033  10.8002   9.6033   9.6033   9.6033
  5  9.6033   9.6033   9.6033   9.6033  10.8002   9.6033   9.6033
  6  9.6033   9.6033   9.6033   9.6033   9.6033  10.8002   9.6033
  7  9.6033   9.6033   9.6033   9.6033   9.6033   9.6033  10.8002


   Estimated R Correlation Matrix for subj(program) 1 RI

Row    Col1     Col2     Col3     Col4     Col5     Col6     Col7

  1  1.0000   0.8892   0.8892   0.8892   0.8892   0.8892   0.8892
  2  0.8892   1.0000   0.8892   0.8892   0.8892   0.8892   0.8892
  3  0.8892   0.8892   1.0000   0.8892   0.8892   0.8892   0.8892
  4  0.8892   0.8892   0.8892   1.0000   0.8892   0.8892   0.8892
  5  0.8892   0.8892   0.8892   0.8892   1.0000   0.8892   0.8892
  6  0.8892   0.8892   0.8892   0.8892   0.8892   1.0000   0.8892
  7  0.8892   0.8892   0.8892   0.8892   0.8892   0.8892   1.0000
```

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
CS	subj(program)	9.6033
Residual		1.1969

Fitting Information

Res Log Likelihood	-710.4
Akaike's Information Criterion	-712.4
Schwarz's Bayesian Criterion	-714.5
-2 Res Log Likelihood	1420.8

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	2	54	3.07	0.0548
time	6	324	7.43	<.0001
program*time	12	324	2.99	0.0005

The Mixed Procedure

Model Information

Data Set	WORK.SET2
Dependent Variable	strength
Covariance Structure	Autoregressive
Subject Effect	subj(program)
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Between-Within

Class Level Information

Class	Levels	Values
program	3	RI WI XCONT
subj	21	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
time	7	2 4 6 8 10 12 14

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	2033.88298	
1	2	1266.80351	0.00000002
2	1	1266.80350	0.00000000

Estimated R Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	10.7600	10.2411	9.7473	9.2772	8.8298	8.4040	7.9988
2	10.2411	10.7600	10.2411	9.7473	9.2772	8.8298	8.4040
3	9.7473	10.2411	10.7600	10.2411	9.7473	9.2772	8.8298
4	9.2772	9.7473	10.2411	10.7600	10.2411	9.7473	9.2772
5	8.8298	9.2772	9.7473	10.2411	10.7600	10.2411	9.7473
6	8.4040	8.8298	9.2772	9.7473	10.2411	10.7600	10.2411
7	7.9988	8.4040	8.8298	9.2772	9.7473	10.2411	0.7600

Estimated R Correlation Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	1.0000	0.9518	0.9059	0.8622	0.8206	0.7810	0.7434
2	0.9518	1.0000	0.9518	0.9059	0.8622	0.8206	0.7810
3	0.9059	0.9518	1.0000	0.9518	0.9059	0.8622	0.8206
4	0.8622	0.9059	0.9518	1.0000	0.9518	0.9059	0.8622
5	0.8206	0.8622	0.9059	0.9518	1.0000	0.9518	0.9059
6	0.7810	0.8206	0.8622	0.9059	0.9518	1.0000	0.9518
7	0.7434	0.7810	0.8206	0.8622	0.9059	0.9518	1.0000

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
AR(1)	subj(program)	0.9518
Residual		10.7600

Fitting Information

Res Log Likelihood	-633.4
Akaike's Information Criterion	-635.4
Schwarz's Bayesian Criterion	-637.4
-2 Res Log Likelihood	1266.8

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	2	54	3.11	0.0528
time	6	324	4.30	0.0003
program*time	12	324	1.17	0.3007

The Mixed Procedure

Model Information

Data Set	WORK.SET2
Dependent Variable	strength
Covariance Structure	Unstructured
Subject Effect	subj(program)
Estimation Method	REML
Residual Variance Method	None
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Between-Within

Class Level Information

Class	Levels	Values
program	3	RI WI XCONT
subj	21	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
time	7	2 4 6 8 10 12 14

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	2033.882984	
1	1	1234.895726	0.00000000

Estimated R Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	8.7804	8.7573	8.9659	8.1986	8.6784	8.2206	8.4172
2	8.7573	9.4732	9.4633	8.5688	9.2015	8.7310	8.6878
3	8.9659	9.4633	10.7083	9.9268	10.6664	10.0704	10.2142
4	8.1986	8.5688	9.9268	10.0776	10.5998	9.8989	10.0436
5	8.6784	9.2015	10.6664	10.5998	12.0954	11.3447	11.3641
6	8.2206	8.7310	10.0704	9.8989	11.3447	11.7562	11.6504
7	8.4172	8.6878	10.2142	10.0436	11.3641	11.6504	12.7104

Estimated R Correlation Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	1.0000	0.9602	0.9246	0.8716	0.8421	0.8091	0.7968
2	0.9602	1.0000	0.9396	0.8770	0.8596	0.8273	0.7917
3	0.9246	0.9396	1.0000	0.9556	0.9372	0.8975	0.8755
4	0.8716	0.8770	0.9556	1.0000	0.9601	0.9094	0.8874
5	0.8421	0.8596	0.9372	0.9601	1.0000	0.9514	0.9165
6	0.8091	0.8273	0.8975	0.9094	0.9514	1.0000	0.9531
7	0.7968	0.7917	0.8755	0.8874	0.9165	0.9531	1.0000

Fitting Information

Res Log Likelihood	-617.4
Akaike's Information Criterion	-645.4
Schwarz's Bayesian Criterion	-674.1
-2 Res Log Likelihood	1234.9

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	2	54	3.07	0.0548
time	6	54	7.12	<.0001
program*time	12	54	1.57	0.1297

Covariance Model	REML log-like.	AIC	SBC
Compound Symmetry (2 parms)	-710.41	-712.41	-716.34
AR(1) (2 parms)	-633.40	-635.40	-639.34
Unstructured (28 parms)	-617.45	-645.44	-700.54

- The AR(1) covariance structure is indicated
- Results may change if the "fixed" part of the model ($X\beta$) is changed

Likelihood ratio tests:

1. H_0: compound symmetry vs. H_A: unstructured

$$(1420.820) - (1234.896) = 185.924$$

on (28)-(2) = 26 d.f.
(p-value = .0001)

2. H_0: AR(1) vs. H_A: unstructured

$$(1266.804) - (1234.896) = 31.908$$

on (28)-(2)=26 d.f.
(p-value = 0.196)

3. Do not use this model to test H_0: AR(1) vs. H_A: compund symmetry


```
                    The Mixed Procedure

                     Model Information

     Data Set                        WORK.SET2
     Dependent Variable              strength
     Covariance Structure            Autoregressive
     Subject Effect                  subj(program)
     Estimation Method               REML
     Residual Variance Method        Profile
     Fixed Effects SE Method         Model-Based
     Degrees of Freedom Method       Between-Within


                  Class Level Information

    Class      Levels    Values

    program         3    RI WI XCONT
    subj           21    1 2 3 4 5 6 7 8 9 10 11 12 13
                         14 15 16 17 18 19 20 21


                     Iteration History

 Iteration    Evaluations    -2 Res Log Like     Criterion

         0              1      2090.993400
         1              2      1293.507056    0.00000122
         2              1      1293.506703    0.00000000
```



```
          Covariance Parameter Estimates

      Cov Parm     Subject          Estimate

      AR(1)        subj(program)      0.9523
      Residual                       10.7585


              Fitting Information

   Res Log Likelihood                   -646.8
   Akaike's Information Criterion       -648.8
   Schwarz's Bayesian Criterion         -650.8
   -2 Res Log Likelihood                1293.5


         Type 1 Tests of Fixed Effects

                       Num     Den
 Effect                 DF      DF    F Value    Pr > F

 program                 2      54       3.10    0.0530
 time                    1     336      12.69    0.0004
 time*program            2     336       4.75    0.0093
 time*time               1     336       7.18    0.0077
 time*time*program       2     336       0.88    0.4167
```



```
                 The Mixed Procedure

                  Model Information

     Data Set                        WORK.SET2
     Dependent Variable              strength
     Covariance Structure            Autoregressive
     Subject Effect                  subj(program)
     Estimation Method               REML
     Residual Variance Method        Profile
     Fixed Effects SE Method         Model-Based
     Degrees of Freedom Method       Between-Within

                  Class Level Information

  Class      Levels    Values

  program         3    RI WI XCONT
  subj           21    1 2 3 4 5 6 7 8 9 10 11 12 13
                       14 15 16 17 18 19 20 21

                     Iteration History

 Iteration    Evaluations    -2 Res Log Like   Criterion

         0              1     2115.710057
         1              2     1323.826724    0.00004384
         2              1     1323.813124    0.00000002
         3              1     1323.813118    0.00000000
```

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
AR(1)	subj(program)	0.9522
Residual		10.7590

Fitting Information

Res Log Likelihood	-661.9
Akaike's Information Criterion	-663.9
Schwarz's Bayesian Criterion	-665.9
-2 Res Log Likelihood	1323.8

Type 1 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	2	54	3.10	0.0530
time	1	333	8.85	0.0031
time*program	2	333	4.39	0.0131
time*time	1	333	7.18	0.0077
time*time*program	2	333	0.88	0.4170
time*time*time	1	333	2.72	0.1001
time*time*time*program	2	333	0.03	0.9740

The Mixed Procedure

Model Information

Data Set	WORK.SET2
Dependent Variable	strength
Covariance Structure	Spatial Power
Subject Effect	subj(program)
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Between-Within

Class Level Information

Class	Levels	Values
program	3	RI WI XCONT
subj	21	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	2090.993400	
1	2	1314.086838	0.11555121
2	1	1293.541779	0.00012430
3	1	1293.506742	0.00000014
4	1	1293.506703	0.00000000

Estimated R Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	10.7584	10.2449	9.7559	9.2902	8.8468	8.4245	8.0224
2	10.2449	10.7584	10.2449	9.7559	9.2902	8.8468	8.4245
3	9.7559	10.2449	10.7584	0.2449	9.7559	9.2902	8.8468
4	9.2902	9.7559	10.2449	10.7584	10.2449	9.7559	9.2902
5	8.8468	9.2902	9.7559	10.2449	10.7584	10.2449	9.7559
6	8.4245	8.8468	9.2902	9.7559	10.2449	10.7584	10.2449
7	8.0224	8.4245	8.8468	9.2902	9.7559	10.2449	10.7584

Estimated R Correlation Matrix for subj(program) 1 RI

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7
1	1.0000	0.9523	0.9068	0.8635	0.8223	0.7831	0.7457
2	0.9523	1.0000	0.9523	0.9068	0.8635	0.8223	0.7831
3	0.9068	0.9523	1.0000	0.9523	0.9068	0.8635	0.8223
4	0.8635	0.9068	0.9523	1.0000	0.9523	0.9068	0.8635
5	0.8223	0.8635	0.9068	0.9523	1.0000	0.9523	0.9068
6	0.7831	0.8223	0.8635	0.9068	0.9523	1.0000	0.9523
7	0.7457	0.7831	0.8223	0.8635	0.9068	0.9523	1.0000

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
SP(POW)	subj(program)	0.9758
Residual		10.7584

Fitting Information

Res Log Likelihood	-646.8
Akaike's Information Criterion	-648.8
Schwarz's Bayesian Criterion	-650.8
-2 Res Log Likelihood	1293.5

Solution for Fixed Effects

Effect	program	Estimate	Standard Error	Pr>|t|
program	RI	78.9054	0.8913	<.0001
program	WI	80.4928	0.7780	<.0001
program	XCONT	79.5708	0.7972	<.0001
time*program	RI	0.4303	0.1315	0.0012
time*program	WI	0.2930	0.1148	0.0111
time*program	XCONT	0.1046	0.1176	0.3746
time*time*program	RI	-0.01942	0.007634	0.0114
time*time*program	WI	-0.00766	0.006664	0.2514
time*time*program	XCONT	-0.00732	0.006828	0.2842

Type 1 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	3	54	12910.9	<.0001
time*program	3	336	7.39	<.0001
time*time*program	3	336	2.98	0.0316

Models for the change in mean strength across time

Program 1: Repetitions increase

$$Y = 78.9054 + 0.4303(\textbf{days}) - .01942(\textbf{days})^2$$
$$(0.8913) \quad (0.1315) \qquad (0.0076)$$

Program 2: Weight increases

$$Y = 80.4928 + 0.2930(\textbf{days}) - .00765(\textbf{days})^2$$
$$(0.7779) \quad (0.1148) \qquad (.00666)$$

Program 3: Controls

$$Y = 79.5708 + 0.1046(\textbf{days}) - .00732(\textbf{days})^2$$
$$(0.7972) \quad (0.1176) \qquad (.00683)$$

The Mixed Procedure

Model Information

Data Set	WORK.SET2
Dependent Variable	strength
Covariance Structure	Unstructured
Subject Effect	subj(program)
Estimation Method	REML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Class Level Information

Class	Levels	Values
program	3	RI WI XCONT
subj	21	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Iteration History

Iteration	Evaluations	-2 Res Log Like	Criterion
0	1	2090.99340	
1	1	1304.01337	0.0000

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	subj(program)	9.3596
UN(2,1)	subj(program)	-0.3437
UN(2,2)	subj(program)	0.1868
UN(3,1)	subj(program)	0.01417
UN(3,2)	subj(program)	-0.00942
UN(3,3)	subj(program)	0.000575
Residual		0.4518

Fitting Information

Res Log Likelihood	-652.0
Akaike's Information Criterion	-659.0
Schwarz's Bayesian Criterion	-666.2
-2 Res Log Likelihood	1304.0

Solution for Fixed Effects

Effect	program	Estimate	Standard Error	Pr>|t|
program	RI	79.0804	0.8084	<.0001
program	WI	80.5238	0.7056	<.0001
program	XCONT	79.6071	0.7231	<.0001
time*program	RI	0.3966	0.1315	0.0039
time*program	WI	0.3005	0.1148	0.0115
time*program	XCONT	0.1116	0.1177	0.3471
time*time*program	RI	-0.01823	0.007547	0.0191
time*time*program	WI	-0.00879	0.006587	0.1878
time*time*program	XCONT	-0.00848	0.006750	0.2143

Type 1 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr > F
program	3	54	12749.5	<.0001
time*program	3	54	6.57	0.0007
time*time*program	3	54	3.06	0.0356

Solution for Random Effects

Effect	prog	subj	Estimate	Std Err Pred	DF	Pr>|t|
Intercept	RI	1	-0.5827	1.1418	228	0.6103
time	RI	1	-0.1998	0.2551	228	0.4343
time*time	RI	1	0.008501	0.01522	228	0.5771
Intercept	RI	2	2.5097	1.1418	228	0.0290
time	RI	2	0.1939	0.2551	228	0.4481
time*time	RI	2	0.003286	0.01522	228	0.8293
Intercept	RI	3	1.4849	1.1418	228	0.1947
time	RI	3	0.04297	0.2551	228	0.8664
time*time	RI	3	-0.00436	0.01522	228	0.7749
.						
.						
.						
Intercept	XC	19	0.4741	1.0937	228	0.6651
time	XC	19	-0.2557	0.2519	228	0.3112
time*time	XC	19	0.01829	0.01507	228	0.2262
Intercept	XC	20	-2.1431	1.0937	228	0.0513
time	XC	20	0.4422	0.2519	228	0.0806
time*time	XC	20	-0.02043	0.01507	228	0.1765


```
# This file posted as  weight2.ssc
#
# This code is applied to the weighting
# data from Littel, et. al. 1991.

temp <- read.table("weight2.dat",
       col.names=c("Subj","Program","S1",
       "S2","S3","S4","S5","S6","S7"))
temp

  Subj Program S1 S2 S3 S4 S5 S6 S7
 1    1   XCONT 85 85 86 85 87 86 87
 2    2   XCONT 80 79 79 78 78 79 78
 3    3   XCONT 78 77 77 77 76 76 77
 4    4   XCONT 84 84 85 84 83 84 85
 5    5   XCONT 80 81 80 80 79 79 80
 6    6   XCONT 76 78 77 78 78 77 74
 .    .       .  .  .  .  .  .  .  .
 .    .       .  .  .  .  .  .  .  .
 .    .       .  .  .  .  .  .  .  .
55   55      WI 85 86 87 86 86 86 86
56   56      WI 77 78 80 81 82 82 82
57   57      WI 80 81 80 81 81 82 83
```



```
# Create a data file where responses at
# different time points are on different
# lines

attach(temp)
weight <- data.frame(rbind(
     cbind(Subj,Program, 2,S1),
     cbind(Subj,Program, 4,S2),
     cbind(Subj,Program, 6,S3),
     cbind(Subj,Program, 8,S4),
     cbind(Subj,Program,10,S5),
     cbind(Subj,Program,12,S6),
     cbind(Subj,Program,Time=14,
                Strength=S7)))
detach( )

# Create factors

weight$Subj <- as.factor(weight$Subj)
weight$Program <- as.factor(weight$Program)
weight$Timef <- as.factor(weight$Time)
```

```
#  Sort the data set by suject

i <- order(weight$Subj,weight$Time)
weight <- weight[i,]

# Delete the list called i
rm(i)

# Compute sample means

means <- tapply(weight$Strength,
    list(weight$Time,weight$Program),mean)
means
         1        2     3
 2 79.6875 81.04762 79.75
 4 80.5625 81.66667 79.95
 6 80.8125 81.90476 80.00
 8 81.0000 82.52381 80.05
10 81.2500 82.61905 79.80
12 81.1250 82.71429 79.60
14 81.1250 83.09524 79.60
```



```
#  Make a profile plot of the means
#  Unix users should insert the motif( )
#  command

#  motif()

x.axis <- unique(weight$Time)

par(fin=c(6.0,6.0),pch=18,mkh=.1,mex=1.5,
            cex=1.2,lwd=3)
matplot(c(2,14), c(79,85.7), type="n",
     xlab="Time(Days)", ylab="Strength",
     main= "Observed Strength Means")
matlines(x.axis,means,type='l',lty=c(1,3,7))
matpoints(x.axis,means, pch=c(16,17,15))
legend(2.1,85.69,legend=c("RI program",
  'WI Program','Controls'),lty=c(1,3,7),bty='n')
```



```
# Use the lme function. This application
# assumes that each subject has a different
# identification value

weight.lme <- lme(Strength ~ Program*Timef,
  random= ~ 1|Subj, data=weight,
  method=c("REML"))
summary(weight.lme)


Linear mixed-effects model fit by REML
 Data: weight
      AIC      BIC    logLik
  774.2068 864.7094 -364.1034

Random effects:
 Formula:  ~ 1 | Subj
        (Intercept) Residual
StdDev:    3.098924 1.094017
```



```
Fixed effects: Strength ~ Program * Timef

                  Value Std.Error  DF   t-value p-value
   (Intercept)  80.94685 0.4170070 324  194.1139  <.0001
      Program1   0.71492 0.5187335  54    1.3782  0.1738
      Program2  -0.56271 0.2901724  54   -1.9392  0.0577
        Timef1   0.28234 0.1031833 324    2.7363  0.0066
        Timef2   0.15390 0.0595729 324    2.5834  0.0102
        Timef3   0.14833 0.0421244 324    3.5212  0.0005
        Timef4   0.09535 0.0326294 324    2.9221  0.0037
        Timef5   0.05080 0.0266418 324    1.9068  0.0574
        Timef6   0.05443 0.0225164 324    2.4172  0.0162
Program1Timef1  -0.06399 0.1283542 324   -0.4985  0.6185
Program2Timef1  -0.09117 0.0717996 324   -1.2698  0.2051
Program1Timef2  -0.02331 0.0741054 324   -0.3146  0.7533
Program2Timef2  -0.05195 0.0414535 324   -1.2532  0.2110
Program1Timef3   0.04229 0.0524004 324    0.8070  0.4203
Program2Timef3  -0.05542 0.0293121 324   -1.8905  0.0596
Program1Timef4   0.00990 0.0405892 324    0.2438  0.8075
Program2Timef4  -0.06142 0.0227050 324   -2.7053  0.0072
Program1Timef5   0.02495 0.0331409 324    0.7529  0.4521
Program2Timef5  -0.05123 0.0185386 324   -2.7636  0.0060
Program1Timef6   0.04503 0.0280092 324    1.6078  0.1089
Program2Timef6  -0.04567 0.0156680 324   -2.9146  0.0038
```

```
anova(weight.lme)

              numDF denDF  F-value p-value
  (Intercept)     1   324 38242.27  <.0001
      Program     2    54     3.07  0.0548
        Timef     6   324     7.37  <.0001
Program:Timef    12   324     2.99  0.0005

#  Use the gls( ) function to fit a
#  model where the errors have a
#  compound symmetry covariance structure
#  within subjects. Random effects are
#  not used to induce correlation.

weight.glscs <- gls(Strength ~ Program*Timef,
  data=weight,
  correlation = corCompSymm(form=~1|Subj),
  method=c("REML"))
summary(weight.glscs)
```



```
Generalized least squares fit by REML
  Model: Strength ~ Program * Timef
  Data: weight
       AIC      BIC    logLik
  774.2068 864.7094 -364.1034

Correlation Structure: Compound symmetry
 Parameter estimate(s):
      Rho
 0.8891824

Coefficients:
                   Value Std.Error  t-value p-value
   (Intercept)  80.94685 0.4170102 194.1124  <.0001
      Program1   0.71492 0.5187376   1.3782  0.1690
      Program2  -0.56271 0.2901746  -1.9392  0.0532
        Timef1   0.28234 0.1031831   2.7363  0.0065
        Timef2   0.15390 0.0595728   2.5834  0.0102
        Timef3   0.14833 0.0421243   3.5212  0.0005
        Timef4   0.09535 0.0326294   2.9221  0.0037
        Timef5   0.05080 0.0266418   1.9068  0.0573
        Timef6   0.05443 0.0225164   2.4172  0.0161
Program1Timef1  -0.06399 0.1283541  -0.4985  0.6184
Program2Timef1  -0.09117 0.0717995  -1.2698  0.2049
Program1Timef2  -0.02331 0.0741053  -0.3146  0.7532
Program2Timef2  -0.05195 0.0414535  -1.2532  0.2109
Program1Timef3   0.04229 0.0524003   0.8070  0.4202
Program2Timef3  -0.05542 0.0293120  -1.8905  0.0595
Program1Timef4   0.00990 0.0405891   0.2438  0.8075
Program2Timef4  -0.06142 0.0227050  -2.7053  0.0071
Program1Timef5   0.02495 0.0331409   0.7529  0.4520
Program2Timef5  -0.05123 0.0185385  -2.7636  0.0060
Program1Timef6   0.04503 0.0280092   1.6078  0.1087
Program2Timef6  -0.04567 0.0156679  -2.9146  0.0038
```



```
anova(weight.glscs)

Denom. DF: 378
              numDF  F-value p-value
  (Intercept)     1 38241.67  <.0001
      Program     2     3.07  0.0478
        Timef     6     7.37  <.0001
Program:Timef    12     2.99  0.0005


# Try an auto regressive covariance
# structures across time within
# subjects

weight.glsar <- gls(Strength ~ Program*Timef,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("REML"))
summary(weight.glsar)
anova(weight.glsar)
```



```
Generalized least squares fit by REML
  Model: Strength ~ Program * Timef
  Data: weight
       AIC      BIC    logLik
  620.1901 710.6927 -287.0951

Correlation Structure: AR(1)
 Parameter estimate(s):
      Phi
 0.9517777

Coefficients:
                   Value Std.Error  t-value p-value
   (Intercept)  80.94685 0.4141946 195.4319  <.0001
      Program1   0.71492 0.5152351   1.3876  0.1661
      Program2  -0.56271 0.2882154  -1.9524  0.0516
        Timef1   0.28234 0.0679388   4.1558  <.0001
        Timef2   0.15390 0.0501479   3.0690  0.0023
        Timef3   0.14833 0.0415677   3.5684  0.0004
        Timef4   0.09535 0.0361776   2.6355  0.0087
        Timef5   0.05080 0.0323638   1.5697  0.1173
        Timef6   0.05443 0.0294717   1.8467  0.0656
Program1Timef1  -0.06399 0.0845121  -0.7571  0.4494
Program2Timef1  -0.09117 0.0472749  -1.9285  0.0545
Program1Timef2  -0.02331 0.0623812  -0.3737  0.7088
Program2Timef2  -0.05195 0.0348952  -1.4888  0.1374
Program1Timef3   0.04229 0.0517079   0.8178  0.4140
Program2Timef3  -0.05542 0.0289247  -1.9158  0.0561
Program1Timef4   0.00990 0.0450030   0.2199  0.8261
Program2Timef4  -0.06142 0.0251740  -2.4400  0.0151
Program1Timef5   0.02495 0.0402588   0.6197  0.5358
Program2Timef5  -0.05123 0.0225202  -2.2750  0.0235
Program1Timef6   0.04503 0.0366611   1.2283  0.2201
Program2Timef6  -0.04567 0.0205077  -2.2268  0.0266
```

```
anova(weight.glsar)

Residual standard error: 3.280263
Degrees of freedom: 399 total; 378 residual
Denom. DF: 378
              numDF  F-value p-value
  (Intercept)     1 39707.11  <.0001
      Program     2     3.27  0.0390
        Timef     6     4.22  0.0004
Program:Timef    12     1.17  0.3000


#  Use an arbitray covariance matrix for
#  observations at different time
#  points within subjects

weight.gls <- gls(Strength ~ Program*Timef,
  data=weight,
  correlation = corSymm(form=~1|Subj),
  weight = varIdent(form = ~ 1|Timef),
  method=c("REML"))
summary(weight.gls)
```



```
Generalized least squares fit by REML
  Model: Strength ~ Program * Timef
  Data: weight
       AIC      BIC    logLik
  640.5287 833.3385 -271.2643

Correlation Structure: General
 Parameter estimate(s):
 Correlation:
     1     2     3     4     5     6
2 0.957
3 0.916 0.935
4 0.858 0.868 0.952
5 0.826 0.848 0.932 0.957
6 0.791 0.815 0.889 0.903 0.948
7 0.778 0.777 0.866 0.879 0.911 0.950

Variance function:
 Structure: Different std. dev. per stratum
 Formula:  ~ 1 | Timef
 Parameter estimates:
 2        4        6        8       10
 1 1.036747 1.099535 1.067836 1.170677
             12       14
        1.158177 1.204893
```



```
Coefficients:
                   Value Std.Error   t-value p-value
   (Intercept)  80.94685 0.4022510  201.2347  <.0001
      Program1   0.71492 0.5003779    1.4288  0.1539
      Program2  -0.56271 0.2799045   -2.0104  0.0451
        Timef1   0.28234 0.0579046    4.8760  <.0001
        Timef2   0.15390 0.0495229    3.1077  0.0020
        Timef3   0.14833 0.0417994    3.5486  0.0004
        Timef4   0.09535 0.0350091    2.7235  0.0068
        Timef5   0.05080 0.0311540    1.6306  0.1038
        Timef6   0.05443 0.0282576    1.9261  0.0548
Program1Timef1  -0.06399 0.0720301   -0.8884  0.3749
Program2Timef1  -0.09117 0.0402927   -2.2627  0.0242
Program1Timef2  -0.02331 0.0616037   -0.3784  0.7053
Program2Timef2  -0.05195 0.0344603   -1.5076  0.1325
Program1Timef3   0.04229 0.0519962    0.8133  0.4166
Program2Timef3  -0.05542 0.0290860   -1.9052  0.0575
Program1Timef4   0.00990 0.0435494    0.2272  0.8204
Program2Timef4  -0.06142 0.0243610   -2.5214  0.0121
Program1Timef5   0.02495 0.0387539    0.6438  0.5201
Program2Timef5  -0.05123 0.0216784   -2.3633  0.0186
Program1Timef6   0.04503 0.0351509    1.2811  0.2009
Program2Timef6  -0.04567 0.0196629   -2.3224  0.0207


Residual standard error: 2.874247
Degrees of freedom: 399 total; 378 residual
```



```
anova(weight.gls)
Denom. DF: 378
              numDF  F-value p-value
  (Intercept)     1 50496.38  <.0001
      Program     2     2.96  0.0528
        Timef     6     6.67  <.0001
Program:Timef    12     1.52  0.1156

#  Compare the fit of various covariance
#  structures.

anova(weight.gls, weight.glscs)

          Model df      AIC      BIC    logLik
  weight.gls   1 49 640.5287 833.3385 -271.2643
weight.glscs   2 23 774.2068 864.7094 -364.1034

                   Test   L.Ratio   p-value
    weight.gls
  weight.glscs   1 vs 2  185.6782   <.0001
```

```
anova(weight.gls, weight.glsar)

          Model df      AIC      BIC    logLik
  weight.gls  1 49 640.5287 833.3385 -271.2643
weight.glsar  2 23 620.1901 710.6927 -287.0951

                   Test    L.Ratio    p-value
   weight.gls
   weight.glsar  1 vs 2   31.66149    0.2046
```



```
# Treat time as a continuous variable and
# fit quadratic trends in strength
# over time

weight.time <- gls(Strength ~ Program+Time+
     Program*Time+Time^2+Program*Time^2,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("REML"))
summary(weight.time)

Generalized least squares fit by REML

  Model: Strength ~ Program + Time +
            Program * Time +
            Time^2 + Program * Time^2
  Data: weight

       AIC     BIC    logLik
  533.0004 576.628 -255.5002

Correlation Structure: AR(1)
 Parameter estimate(s):
       Phi
 0.9522699
```



```
Coefficients:
                      Value Std.Error   t-value p-value
      (Intercept)  79.65634 0.4755150  167.5159  <.0001
             Time   0.27597 0.0701659    3.9332  0.0001
         Program1   0.79369 0.5915142    1.3418  0.1804
         Program2  -0.04277 0.3308849   -0.1293  0.8972
        I(Time^2)  -0.01147 0.0040732   -2.8149  0.0051
     TimeProgram1  -0.06864 0.0872825   -0.7864  0.4321
     TimeProgram2  -0.08570 0.0488246   -1.7552  0.0800
Program1I(Time^2)   0.00588 0.0050668    1.1605  0.2466
Program2I(Time^2)   0.00207 0.0028343    0.7305  0.4655

Residual standard error: 3.28002
Degrees of freedom: 399 total; 390 residual

anova(weight.time)

Denom. DF: 390
                  numDF  F-value p-value
      (Intercept)     1 39659.09  <.0001
             Time     1    12.69  0.0004
          Program     2     3.27  0.0391
        I(Time^2)     1     7.18  0.0077
     Program:Time     2     4.75  0.0092
Program:I(Time^2)     2     0.88  0.4166
```



```
# To compare the continuous time model to the
# model where we fit a different mean at each
# time point, we must compare likelihood values
# instead of REML likelihood values.

weight.glsarmle <- gls(Strength ~ Program*Timef,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))
summary(weight.glsar)

Generalized least squares fit by REML
  Model: Strength ~ Program * Timef
  Data: weight
       AIC      BIC    logLik
  620.1901 710.6927 -287.0951

Correlation Structure: AR(1)
 Parameter estimate(s):
       Phi
 0.9517777
```

```
Coefficients:
                   Value Std.Error   t-value p-value
   (Intercept)  80.94685 0.4141946  195.4319  <.0001
      Program1   0.71492 0.5152351    1.3876  0.1661
      Program2  -0.56271 0.2882154   -1.9524  0.0516
        Timef1   0.28234 0.0679388    4.1558  <.0001
        Timef2   0.15390 0.0501479    3.0690  0.0023
        Timef3   0.14833 0.0415677    3.5684  0.0004
        Timef4   0.09535 0.0361776    2.6355  0.0087
        Timef5   0.05080 0.0323638    1.5697  0.1173
        Timef6   0.05443 0.0294717    1.8467  0.0656
Program1Timef1  -0.06399 0.0845121   -0.7571  0.4494
Program2Timef1  -0.09117 0.0472749   -1.9285  0.0545
Program1Timef2  -0.02331 0.0623812   -0.3737  0.7088
Program2Timef2  -0.05195 0.0348952   -1.4888  0.1374
Program1Timef3   0.04229 0.0517079    0.8178  0.4140
Program2Timef3  -0.05542 0.0289247   -1.9158  0.0561
Program1Timef4   0.00990 0.0450030    0.2199  0.8261
Program2Timef4  -0.06142 0.0251740   -2.4400  0.0151
Program1Timef5   0.02495 0.0402588    0.6197  0.5358
Program2Timef5  -0.05123 0.0225202   -2.2750  0.0235
Program1Timef6   0.04503 0.0366611    1.2283  0.2201
Program2Timef6  -0.04567 0.0205077   -2.2268  0.0266

Residual standard error: 3.280263
Degrees of freedom: 399 total; 378 residual
```



```
weight.timemle <- gls(Strength ~ Time+
     Program*Time+Time^2+Program*Time^2,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))

anova(weight.glsarmle, weight.timemle)

            Model df      AIC      BIC    logLik
weight.glsarmle 1 23 484.9287 576.6749 -219.4644
 weight.timemle 2 11 469.8739 513.7524 -223.9369
                   Test     L.Ratio   p-value
 weight.glsarmle
  weight.timemle  1 vs 2   8.945125  0.7076
```



```
#  Do not fit different quadratic trends
#  for different programs

weight.timemle <- gls(Strength ~ Time+
     Program*Time+Time^2, data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))

anova(weight.glsarmle, weight.timemle)
             Model df     AIC     BIC    logLik
weight.glsarmle  1 23 484.928 576.675 -219.4644
 weight.timemle  2  9 467.653 503.554 -224.8267

                  Test   L.Ratio   p-value
weight.glsarmle
 weight.timemle  1 vs 2   10.7247   0.7075
```



```
# Treat time as a continuous variable and
# fit quadratic trends in strength
# over time

weight.time <- gls(Strength ~ Program+ Time+
     Program*Time+Time^2+Program*Time^2,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("REML"))
summary(weight.time)

Generalized least squares fit by REML
  Model: Strength ~ Program + Time +
                    + Program * Time
                    + Time^2 + Program * Time^2
  Data: weight
      AIC     BIC    logLik
  533.0004 576.628 -255.5002

Correlation Structure: AR(1)
 Parameter estimate(s):
      Phi
 0.9522699
```

```
Coefficients:
                      Value Std.Error   t-value p-value
      (Intercept)  79.65634 0.4755150  167.5159  <.0001
         Program1   0.79369 0.5915142    1.3418  0.1804
         Program2  -0.04277 0.3308849   -0.1293  0.8972
             Time   0.27597 0.0701659    3.9332  0.0001
        I(Time^2)  -0.01147 0.0040732   -2.8149  0.0051
     Program1Time  -0.06864 0.0872825   -0.7864  0.4321
     Program2Time  -0.08570 0.0488246   -1.7552  0.0800
Program1I(Time^2)   0.00588 0.0050668    1.1605  0.2466
Program2I(Time^2)   0.00207 0.0028343    0.7305  0.4655


anova(weight.time)

Denom. DF: 390
                  numDF  F-value p-value
      (Intercept)     1 39659.09  <.0001
          Program     2     3.27  0.0391
             Time     1    12.69  0.0004
        I(Time^2)     1     7.18  0.0077
     Program:Time     2     4.75  0.0092
Program:I(Time^2)     2     0.88  0.4166
```



```
# To compare the continuous time model to the
# model where we fit a different mean at each
# time point, we must compare likelihood values
# instead of REML likelihood values.


weight.glsarmle <- gls(Strength ~ Program*Timef,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))


weight.timemle <- gls(Strength ~ Program+ Time+
     Program*Time+Time^2+Program*Time^2,
  data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))
```



```
anova(weight.glsarmle, weight.timemle)

              Model df     AIC     BIC    logLik
weight.glsarmle   1 23 484.929 576.675 -219.4644
 weight.timemle   2 11 469.874 513.752 -223.9369

                    Test  L.Ratio  p-value
weight.glsarmle
 weight.timemle    1 vs 2 8.9451250.7076
```



```
#  Do not fit different quadratic trends
#  for different programs

weight.timemle <- gls(Strength ~ Program + Time+
     Program*Time+Time^2, data=weight,
  correlation = corAR1(form=~1|Subj),
  method=c("ML"))

anova(weight.glsarmle, weight.timemle)
                Model df     AIC     BIC   logLik
weight.glsarmle     1 23 484.929 576.675 -219.464
 weight.timemle     2  9 467.653 503.554 -224.827

                   Test   L.Ratio  p-value
weight.glsarmle
 weight.timemle  1 vs 2   10.724   0.7075
```

```
> weight.timer <- lme(Strength ~ Program + Time+
+      Program*Time+Time^2,
+   random = ~ Time + Time^2 | Subj,
+   data=weight,
+   correlation = corAR1(form=~1|Subj),
+   method=c("REML"))
> summary(weight.timer)

Linear mixed-effects model fit by REML
 Data: weight
       AIC     BIC  logLik
  514.0401 573.609 -242.02

Random effects:
 Formula:  ~ Time + Time^2 | Subj
 Structure: General positive-definite
                 StdDev   Corr
(Intercept) 2.444260999 (Intr) Time
       Time 0.089532624  0.911
  I(Time^2) 0.006017243 -0.442 -0.447
   Residual 1.436471358

Correlation Structure: AR(1)
 Parameter estimate(s):
       Phi
 0.7626235
```



```
Fixed effects: Strength ~ Program + Time +
                          Program * Time + Time^2

                 Value Std.Error  DF   t-value p-value
 (Intercept)  79.68923 0.4184179 338  190.4537  <.0001
    Program1   0.61624 0.4996127  54    1.2334  0.2228
    Program2  -0.10485 0.2794765  54   -0.3752  0.7090
        Time   0.26482 0.0663909 338    3.9888  0.0001
   I(Time^2)  -0.01088 0.0039624 338   -2.7449  0.0064
Program1Time   0.02358 0.0298548 338    0.7897  0.4303
Program2Time  -0.05327 0.0167004 338   -3.1900  0.0016


> anova(weight.timer)
             numDF denDF  F-value p-value
 (Intercept)     1   338 41957.34  <.0001
     Program     2    54     1.57  0.2178
        Time     1   338    15.61  0.0001
   I(Time^2)     1   338     7.53  0.0064
Program:Time     2   338     5.64  0.0039
```



```
>  ranef(weight.timer)

   (Intercept)         Time      I(Time^2)
 1  4.37055391  0.153001109 -0.00158224377
 2 -0.65665935 -0.027283603 -0.00049600632
 3 -1.99187071 -0.071099280  0.00228432575
 4  3.21280216  0.110654243 -0.00132097835
 5  0.06308721  0.001557145  0.00045597473
 6 -2.48768170 -0.085879558 -0.00162573342
 7 -0.19374915 -0.002685795  0.00334519697
 8 -3.37164937 -0.119067097  0.00149650511
 9 -0.80881925 -0.018877607  0.00383983997
10 -0.82383506 -0.028933723  0.00178975990
11  0.77254851  0.027237940  0.00186592043
12 -2.08399741 -0.070464612  0.00269082474
13  2.17945957  0.077251958 -0.00134446211
14  1.48834972  0.037057092 -0.00844304252
15  0.48471173  0.021536591 -0.00092444374
16 -1.18425615 -0.043488611  0.00130028422
17  2.35410656  0.079137338 -0.00373376183
18 -1.39497258 -0.048915776  0.00006527998
19 -0.02396794 -0.001134037  0.00099089967
20 -0.40816670 -0.008283625  0.00215175351
21 -0.89039491 -0.032206156  0.00088770301
22  3.17654453  0.114036836 -0.00025741770
23  1.09498162  0.036964012 -0.00239528917
24  0.65881585  0.019911728 -0.00253575809
25  1.23442866  0.051135953  0.00466894856
26 -3.44574955 -0.122468721  0.00071839368
27  2.00624995  0.074780559 -0.00002104473
28 -1.02638515 -0.027887317  0.00349291296
29  3.89421553  0.135196521 -0.00501762284
30 -2.92889483 -0.091726521  0.00446638045
```



```
31 -3.01095558 -0.105261187  0.00091354008
32  3.36154446  0.115634836 -0.00290042981
33 -0.27155045 -0.006258083  0.00260864631
34 -2.75884302 -0.103631969 -0.00002245946
35 -2.66199251 -0.101652914 -0.00238218115
36  2.55089681  0.079861940 -0.00769636867
37  1.28587092  0.037557863 -0.00298425029
38 -5.16208816 -0.175294866  0.00510901756
39  0.40620756  0.006347161 -0.00342714273
40  3.27227403  0.107103127 -0.00621986022
41  1.29362782  0.048333813  0.00084250740
42 -1.97591192 -0.070223269  0.00121647507
43 -1.15430424 -0.034999541  0.00419417553
44  5.62624670  0.198899755 -0.00335004655
45 -0.23229246 -0.008378923  0.00079947844
46  1.26600824  0.049098060  0.00302455004
47 -1.48093721 -0.050034234  0.00145338926
48 -0.90391122 -0.027275502  0.00123120483
49  0.99200758  0.029291973 -0.00378712621
50  0.26440449  0.012397816  0.00261403536
51 -2.64122541 -0.092388808  0.00176277381
52  0.81701402  0.024595675 -0.00107110252
53 -1.59952829 -0.059437331  0.00003655025
54 -0.78800413 -0.028632941 -0.00127495212
55  2.74460819  0.091004189 -0.00440315772
56 -1.74908270 -0.051129697  0.00430386546
57 -0.75988926 -0.024583931  0.00259576899
```

```
> weight.timeru <- lme(Strength ~ Program + Time+
+      Program*Time+Time^2,
+   random = ~ Time + Time^2 | Subj,
+   data=weight,
+   method=c("REML"))
>
> anova(weight.timer,weight.timeru)
              Model df     AIC     BIC    logLik
 weight.timer     1 15 514.040 573.609 -242.0200
weight.timeru     2 14 527.917 583.514 -249.9583

                 Test    L.Ratio   p-value
 weight.timer
weight.timeru  1 vs 2  15.87654    0.0001
```