1.(a) Since A is an orthogonal matrix, we have $A^TA = I$. From the properties of determinants (page 49 of the course notes) we have $|A| = |A^T|$ and $|A^TA| = |A^T||A|$. It follows that $|A|^2 = |A||A| = |A^T||A| = |A^TA| = |I| = 1$. Hence, $|A| = 1$ or $|A| = -1$.

(b) Since an idempotent matrix satisfies $AA = A$, we have $|A| = |AA| = |A||A| = |A|^2$. Consequently, $|A| = 1$ or $|A| = 0$.

2. Because B is a $k \times p$ matrix of rank k, the rows of B are a set of linearly independent vectors. This implies that $\underline{w} = B^T\underline{y} \neq \underline{0}$ for any $\underline{y} \neq \underline{0}$. Then, since A is positive definite, $\underline{y}^TBAB^T\underline{y} = \underline{w}^TA\underline{w} > 0$ for any $\underline{y} \neq \underline{0}$. Therefore, BAB^T is positive definite.

3. First show that if $A = P^TP$ for some nonsingular matrix P, then A is positive definite and symmetric.

Suppose $A = P^TP$ for some nonsingular matrix P. Then for any $\underline{y} \neq \underline{0}$, we have $\underline{y}^TA\underline{y} = \underline{y}^TP^TP\underline{y} = (P\underline{y})^TP\underline{y}$. Since P is nonsingular and $\underline{y} \neq \underline{0}$, it follows that $P\underline{y} \neq \underline{0}$. Consequently, $\underline{y}^TA\underline{y} = (P\underline{y})^TP\underline{y} > 0$ for any $\underline{y} \neq \underline{0}$. Therefore, A is positive definite. It follows immediately from $A = P^TP$ that A is symmetric.

Now show if A is a symmetric and positive definite matrix, then $A = P^TP$ for some nonsingular matrix P.

Suppose A is a symmetric, positive definite matrix. Then, we can use the spectral decomposition of A to obtain $A = UDU^T$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_P \end{bmatrix},$$

is a diagonal matrix with the eigenvalues of A on the diagonal and the i-*th* column of U is the eigenvector corresponding to λ_i. Since A is positive definite, all of the eigenvalues are positive and we can define

$$D = D^{1/2}(D^{1/2})^T, \quad \text{where} \quad D^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_P} \end{bmatrix}.$$

Let $P^T = UD^{1/2}$, then we have $A = P^TP$. Furthermore, since A is positive definite $\underline{y}^TA\underline{y} > 0$ for any $\underline{y} \neq \underline{0}$. Then, $\underline{y}^TA\underline{y} = \underline{y}^TP^TP\underline{y} > 0$ which implies that $P\underline{y} \neq \underline{0}$ for all $\underline{y} \neq \underline{0}$. Consequently, P is nonsingular.

4. First create a matrix in S-Plus.

```
> V<-matrix(c(5.0,4.0,3.2,4.0,5.0,4.0,3.2,4.0,5.0),ncol=3)
> V
     [,1] [,2] [,3]
[1,]  5.0    4  3.2
[2,]  4.0    5  4.0
[3,]  3.2    4  5.0
```

(a) Evaluate eigenvalues and eigenvectors

```
> eigen(V)
$values:
[1] 12.4787754  1.8000000  0.7212246
```

```
$vectors:
          [,1]          [,2]       [,3]
[1,] 0.5639516 -7.071068e-001  0.4265661
[2,] 0.6032555  5.872988e-016 -0.7975480
[3,] 0.5639516  7.071068e-001  0.4265661
```

(b) Evaluate the trace of V:

```
> sum(diag(V))
[1] 15
```

(c) Evaluate the determinant of V

```
> prod(eigen(V)$values)
[1] 16.2
```

(d) Evaluate the inverse of V

```
> solve(V)
               [,1]       [,2]           [,3]
[1,]  5.555556e-001 -0.4444444 -2.094729e-016
[2,] -4.444444e-001  0.9111111 -4.444444e-001
[3,] -6.317476e-017 -0.4444444  5.555556e-001
```

5. (a)

$$\text{If } \; B = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_n} \end{bmatrix} \quad \text{then } \; B\Sigma B \; \text{ is a correlation matrix.}$$

(b) Here is one possible way to construct a function to evaluate correlation matrices.

```
> corr.matrix <- function(x){
+      a<-diag( diag(x)^(-0.5) )
+      r<-a%*%x%*%a
+      return(r) }
```

(c) Compute a correlation matrix from the V matrix in problem 6.

```
> corr.matrix(V)
> corr.matrix(V)
     [,1] [,2] [,3]
[1,] 1.00  0.8 0.64
[2,] 0.80  1.0 0.80
[3,] 0.64  0.8 1.00
```

6. (a) Compute determinants of A and B:

```
> A<-matrix(c(4,4.001,4.001,4.002),ncol=2)
> B<-matrix(c(4,4.001,4.001,4.002001),ncol=2)

> prod(eigen(A)$values)
[1] -1e-006

> prod(eigen(B)$values)
[1] 3e-006
```

(b) This is an example of a situation the solve( ) function fails to find the inverse of nearly singular matrices. It was put on this assignment to help you gain a healthy respect for round off errors and the inability of any computer to store numbers with infinite precision. You could easily compute the inverses of A and B without using a computer. If you want to use S-Plus 6 for Windows, you can use the ginverse function, which can compute an inverse or a generalized inverse of a matrix. Note the difference in the inverses, even though A and B are nearly the same.

```
> ginverse(A)
         [,1]     [,2]
[1,] -4002000  4001000
[2,]  4001000 -4000000
attr(, "rank"):
[1] 2

> ginverse(B)
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
attr(, "rank"):
[1] 2
```

In S-Plus version 5 for UNIX or S-Plus 2000 for windows, you can use the command "library(Matrix)" to establish a link to the Matrix library of functions. The Matrix library comes with the older versions of S-Plus and is automatically installed when S-Plus is installed on a computer. It is a library of functions that perform matrix operations with a higher level of precision than the functions in the default library. The solve.Matrix function in this library can compute the inverse of these two nearly singular matrices. The Matrix library of functions is not available in S-Plus 6 for Windows.

```
> library(Matrix)
> solve.Matrix(A)
         [,1]     [,2]
[1,] -4002000  4001000
[2,]  4001000 -4000000
attr(, "class"):
[1] "Matrix"

> solve.Matrix(B)
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
attr(, "class"):
[1] "Matrix"
```

7. (a)

$$R^{-1} = \begin{bmatrix} \frac{1}{1-b^2} & \frac{-b}{1-b^2} \\ \frac{-b}{1-b^2} & \frac{1}{1-b^2} \end{bmatrix}$$

(b)

$$R^{-1} = \begin{bmatrix} \frac{1}{1-b^2} & \frac{-b}{1-b^2} & 0 \\ \frac{-b}{1-b^2} & \frac{1+b^2}{1-b^2} & \frac{-b}{1-b^2} \\ 0 & \frac{-b}{1-b^2} & \frac{1}{1-b^2} \end{bmatrix}$$

(c) By numerically obtaining the inverses of 4×4 and 5×5 versions of R, it is readily seen that the pattern suggested by the results in parts (a) and (b) holds for any order. For an nxn matrix of this type, the inverse of R is a tri-diagonal matrix of the form

$$R^{-1} = \begin{cases} \frac{1}{1-b^2} & i = j = 1 \text{ or } i = j = n, \\ \frac{1+b^2}{1-b^2} & i = j = 2, \ldots, n-1, \\ \frac{-b}{1-b^2} & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the inverse of V is a tri-diagonal matrix of the form

$$V^{-1} = \begin{cases} \frac{1}{\sigma^2(1-b^2)} & i = j = 1 \text{ or } i = j = n, \\ \frac{1+b^2}{\sigma^2(1-b^2)} & i = j = 2, \ldots, n-1, \\ \frac{-b}{\sigma^2(1-b^2)} & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$