Formula Sheet

Selected matrix formulas:

$(AB)^{-1} = B^{-1}A^{-1}$	when A and B are square and non-singular
$(A^T)^{-1} = (A^{-1})^T$	when A is square and nonsingular
$(cA)^{-1} = \frac{1}{c}A^{-1}$	when c is a scalar
$\text{tr }(cA) = c \text{ tr }(A)$	when c is a scalar
$\text{tr }(A+B) = \text{tr }(A) + \text{ tr }(B)$	
$\text{tr }(AB) = \text{tr }(BA)$	
$\lvert A^T\rvert = \lvert A\rvert$	
$\lvert AB\rvert = \lvert A\rvert\,\lvert B\rvert$	for square matrices of the same order
$\lvert AB\rvert = \lvert BA\rvert$	
$\lvert cA\rvert = c^k\lvert A\rvert$	when c is a scalar and A is a $k \times k$ matrix
$A = \sum_{i=1}^{k} \lambda_i \mathbf{u}_i \mathbf{u}_i^T$	spectral decomposition of a symmetric matrix
$A^{-1/2} = \sum_{i=1}^{k} \lambda_i^{-1/2} \mathbf{u}_i \mathbf{u}_i^T$	
$E(A\mathbf{Y} + \mathbf{d}) = A\boldsymbol{\mu} + \mathbf{d}$	when $E(\mathbf{Y}) = \boldsymbol{\mu}$
$Var(A\mathbf{Y} + \mathbf{d}) = A\Sigma A^T$	when $Var(\mathbf{Y}) = \Sigma$
$Cov(A\mathbf{Y} + d,\ \ B\mathbf{Y} + \mathbf{L}) = A\Sigma B^T$	when $Var(\mathbf{Y}) = \Sigma$
$(-1)^{i+j}\lvert A_{ji}\rvert / \lvert A\rvert$	is the $s(i,\ j)$ element of A^{-1},where A_{ji} is the minor of the $(j,\ i)$ element of A
$AA^{-}A = A$	generalized inverse
$AMA = A$ $MAM = M$ AM and MA are symmetric	Moore-Penrose generalized inverse
$P_X = X(X^TX)^{-}X^T$	Projection matrix: You may use the facts that projection matrices are idempotent and symmetric, P_X is invariant to the choice of $(XTX)^{-}$,
$P_X X = X$ $\text{rank}(P_X) = \text{rank}(X) = tr(P_X)$	
$P_X \mathbf{V} = \mathbf{V}$	for any vector $\mathbf{V}$ that is a linear combination of the columns of X.
$P_X \mathbf{V} = \mathbf{O}$	for any $\mathbf{V}$ that is orthogonal to every column of X

Linear models: $\mathbf{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}$

$E(\mathbf{Y}) = X\boldsymbol{\beta}$ $\qquad$ $Var(\mathbf{Y}) = \Sigma$

$(X^TX)\mathbf{b} = X^T\mathbf{Y}$ $\qquad$ normal equations

$(X^T\Sigma^{-1}X)\mathbf{b} = X^T\Sigma^{-1}\mathbf{Y}$ $\qquad$ generalized least square estimating equations.

Estimable functions of parameters: $C\boldsymbol{\beta}$ is estimable if $E(A\mathbf{Y}) = C\boldsymbol{\beta}$ for some A.

Conditions you can check (Result 3.9) are:

(i) $C = AX$ for some A

(ii) $C\mathbf{d} = \mathbf{0}$ for every $\mathbf{d}$ for which $X\mathbf{d} = \mathbf{0}$

Reparameterization: $X = WG$ and $W = XF$

Quadratic forms: $\mathbf{Y}^TA\mathbf{Y}$

$E(\mathbf{Y}^TA\mathbf{Y}) = \boldsymbol{\mu}^TA\boldsymbol{\mu} + tr(A\Sigma)$ $\qquad$ when $E(Y) = \boldsymbol{\mu}$ and $Var(\mathbf{Y}) = \Sigma$

$Var(\mathbf{Y}^TA\mathbf{Y}) = 4\boldsymbol{\mu}^TA\Sigma A\boldsymbol{\mu} + 2tr(A\Sigma A\Sigma)$ $\qquad$ when A is symmetric and $\mathbf{Y} \sim N(\boldsymbol{\mu},\, \Sigma)$

Result 4.7 If $\mathbf{Y} \sim N(\boldsymbol{\mu},\, \Sigma)$ where Σ is positive definite, and if A is symmetric with rank $(A) = k$, and $A\Sigma$ is idempodent, then

$$\mathbf{Y}^TA\mathbf{Y} \;\sim\; \chi^2_k(\boldsymbol{\mu}^TA\boldsymbol{\mu})$$

Result 4.8 If $\mathbf{Y} \sim N(\boldsymbol{\mu},\, \Sigma)$ and A_1 and A_2 are symmetric matrices such that $A_1\Sigma A_2 = 0$, then $\mathbf{Y}^TA_1\mathbf{Y}$ and $\mathbf{Y}^TA_2\mathbf{Y}$ are independent random variables.

Cochran's Theorem: If $\mathbf{Y} \sim N(\boldsymbol{\mu},\, \sigma^2 I)$ and $A_1,\, A_2,\, \ldots,\, A_k$ are $n \times n$ symmetric matrices such that $I = A_1 + A_2 + \ldots + A_k$ and $n = \text{rank}(A_1) + \text{rank}(A_2) + \ldots + \text{rank}(A_k)$, then $\mathbf{Y}^TA_1\mathbf{Y},\, \ldots, \mathbf{Y}^TA_k\mathbf{Y}$ are independent random variables, with $\mathbf{Y}^TA_i\mathbf{Y} \sim \chi^2_{\text{rank}(A_i)}(\frac{1}{\sigma^2}\boldsymbol{\mu}^TA_i\boldsymbol{\mu})$

Tests of hypothesis and confidence intervals:

$H_0:\; C\boldsymbol{\beta} = \mathbf{d}$ $\qquad$ vs $\qquad$ $A_a: C\boldsymbol{\beta} \neq \mathbf{d}$

$$SS_{H0} = (C\mathbf{b} - \mathbf{d})^T[C(XTX)^-C^T]^{-1}(C\mathbf{b} - \mathbf{d})$$

$$F = \frac{MS_{H0}}{MS_{\text{residuals}}}$$

$$t = \frac{\mathbf{c}^T\mathbf{b} - \mathbf{c}^T\boldsymbol{\beta}}{\sqrt{MS_{\text{residuals}}\mathbf{c}^T(XTX)^-\mathbf{c}}}$$

Things you should know, but are not on the formula sheet:

1. What it means for a matrix to be orthogonal, non-singular, positive definite, non-negative definite, idempodent, symmetric.
2. Definition of eigenvalues and eigenvectors.
3. Concepts of rank of a matrix, linearly independent vectors, basis for a vector space, what is means for a set of vectors to span a vector space.
4. How to obtain a generalized inverse.
5. Properties of ordinary least squares estimators.
6. Properties of generalized least squares estimators.
7. Gauss-Markov Theorem.
8. Any linear function of a multivariate normal random variable has a multivariate normal distribution.
9. If $\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}$ has a multivariate normal distribution, then $\mathbf{Y}_1$ is stochastically independent of $\mathbf{Y}_2$ if and only if $Cov(\mathbf{Y}_1,\ \mathbf{Y}_2) = 0$.
10. Definitions of chi-square, F, and t distributions.
11. The definition of a testable hypothesis.
12. Concepts of Type I error, Type II error, and power.
13. How to construct confidence intervals for estimable functions of $\mathbf{B}$ for the normal-theory Gauss-Markov model.
14. How to construct a confidence interval for a variance.
15. How to partition sums of squares.
16. How to interpret S-PLUS or SAS/IML commands for performing computations involving matrices.