Math 166 Midterm - Solutions

4 October 2007

SECTION 1: NO CALCULATOR.

1. Use integration by parts to express $\int (\ln x)^n \, dx$ in terms of $\int (\ln x)^{n-1} \, dx$.

Solution. We follow the directions, taking $u = (\ln x)^n$ and $dv = dx$. Then $du = n \cdot (\ln x)^{n-1} \cdot \frac{1}{x} \, dx$, and $v = x$. Then the integration by parts formula gives

$$\int (\ln x)^n \, dx = x(\ln x)^n - \int x \cdot n \cdot (\ln x)^{n-1} \cdot \frac{1}{x} \, dx = x(\ln x)^n - n \int (\ln x)^{n-1} \, dx,$$

as desired. □

2. Evaluate $\int \sin^3 x \cos^2 x \, dx$.

Solution. We make use of the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$. Hence $\frac{du}{dx} = -\sin x$, so

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin x \sin^2 x \cos^2 x \, dx \\
&= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx \\
&= -\int (1 - u^2) u^2 \, du \\
&= -\int u^2 - u^4 \, du \\
&= u^5/5 - u^3/3 + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

□

3. Evaluate $\int (x^2 + 4)^{-3/2} \, dx$.

Solution. We use the substitution $2 \tan u = x$, so $2 \sec^2 u \, du = dx$. Then, by using the Pythagorean identity $(\tan^2 x + 1 = \sec^2 x)$, we have

$$\begin{aligned}
\int \frac{dx}{(x^2 + 4)^{3/2}} &= \int \frac{2 \sec^2 u}{(4 \tan^2 u + 4)^{3/2}} \, du \\
&= \int \frac{2 \sec^2 u}{(4 \sec^2 u)^{3/2}} \, du \\
&= \frac{1}{4} \int \frac{\sec^2 u}{\sec^3 u} \, du \\
&= \frac{1}{4} \int \frac{1}{\sec u} \, du \\
&= \frac{1}{4} \int \cos u \, du \\
&= \frac{1}{4} \sin u + C \\
&= \frac{1}{4} \sin(\arctan(x/2)) + C \\
&= \frac{1}{4} \cdot \frac{x}{\sqrt{x^2 + 4}} + C.
\end{aligned}$$

□

4. Evaluate $\int \frac{2x^2 - 5x}{(x+2)(x-1)^2}\, dx$.

Solution. We proceed by the method of partial fractions. Since we have one distinct linear factor and one repeated linear factor, we wish to find real numbers A, B, and C such that

$$\frac{2x^2 - 5x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}. \tag{1}$$

Equation (1) holds if and only if

$$2x^2 - 5x = A(x-1)^2 + B(x+2)(x-1) + C(x+2), \tag{2}$$

which holds if and only if the three equations

$$A + B = 2, -2A + B + C = 5, A - 2B + 2C = 0.$$

(The above equations come from comparing coefficients in (2).) Hence $A = 2, B = 0$, and $C = -1$. Then (1) gives

$$\begin{aligned} \int \frac{2x^2 - 5x}{(x+2)(x-1)^2}\, dx &= \int \frac{2}{x+2} - \frac{1}{(x-1)^2}\, dx \\ &= \ln(x+2) + (x-1)^{-1} + C. \end{aligned}$$

□

Section 2: No calculator.

1. Find the area of the region bounded by the curve $y = 3(x^3 - x)$ and the x-axis.

Solution. Since $y = 3(x^3 - x)$ is an odd function (hence symmetric about the origin) and intersects the x-axis at the points $x = -1$ and $x = 1$, the total area bound by the curve and the x-axis is given by $2\int_{-1}^{0} 3(x^3 - x)\, dx$. Then

$$2\int_{-1}^{0} 3(x^3 - x)\, dx = 6\Big|_{-1}^{0} x^4/4 - x^2/2 = 6(0 - 1/4 + 1/2) = 3/2.$$

□

2. Find the length of the curve $y = 2x^{3/2}$ between $x = \frac{1}{3}$ and $x = \frac{8}{3}$.

Solution. This curve is given by the parametric equations $x = t$, $y = 2t^{3/2}$ for $1/3 \leq t \leq 8/3$. Then $x' = 1$ and $y = 3t^{1/2}$. Since x' is nowhere zero, x' and y' cannot be simultaneously zero, so the curve is smooth. Then the length of the curve is given by

$$\int_{1/3}^{8/3} \sqrt{(x')^2 + (y')^2}\, dt = \int_{1/3}^{8/3} \sqrt{1 + 9t}\, dt.$$

Using the substitution $u = 1 + 9t$ (so $du = 9dt$), we have that $u = 25$ when $t = 8/3$ and $u = 4$ when $t = 1/3$. Then

$$\begin{aligned}\int_{1/3}^{8/3} \sqrt{1+9t}\, dt &= \frac{1}{9}\int_4^{25} \sqrt{u}\, du \\ &= \frac{2}{27}\Big|_4^{25} u^{3/2} \\ &= \frac{2}{27}(25^{3/2} - 4^{3/2}) \\ &= 26/3,\end{aligned}$$

which, ironically, is not equal to 8.6667. □

3. Sketch the region R bounded by $y = 2x$, the x-axis, and $x = 1$. Set up (but do not evaluate) an integral for the volume of each solid generated by revolving the region about

   (a) the line $y = 3$.

   (b) the line $x = 3$.

   Solution for (a). Cutting our strips vertically, we see that our strips will sweep out washers (disks with holes cut out of them). To compute the volume of such an object, we will compute the volume of the washer as though it did not have a hole in it, and subtract the volume of the hole. The thickness of an arbitrary washer will be Δx (because our strips have width in the x direction). Since the outer radius is constant (3), the volume of an arbitrary washer (with no hole) is 9π. Since the inner radius is $3 - 2x$, the volume of an arbitrary hole is $\pi(3 - 2x)^2$. Since our strips start at $x = 0$ and end at $x = 1$, the volume of the whole solid is $\int_0^1 9\pi - \pi(3 - 2x)^2\, dx$. □

   Solution for (b). We use vertical strips again, which will sweep out cylindrical shells with thickness Δx. The radius of an arbitrary cylindrical shell is $3 - x$, and the height is $y = 2x$. Thus the volume of an arbitrary shell is $2\pi(3 - x)2x\Delta x$, and hence the volume of the whole solid is $\int_0^1 2\pi(3 - x)2x\, dx$. □

4. A water tank is in the shape of a right circular cone, with the vertex at the top, and the circular base at the bottom. The height of the tank is 10 feet, and the radius at the base is 20 feet.

   If the tank is initially full of water, how much work (in foot-pounds) is needed to pump all of the water out through a hole in the top of the tank? (The density of water is $\delta = 62.4$ pounds per cubic foot.)

   Solution. We begin by embedding a vertical cross section of the conical tank in the plane, with vertex at the point $(0, 10)$. We compute the total work required to pump the water out of the tank by cutting the cross section into horizontal strips and computing the work required for an arbitrary slab of water.

   Since the tank is a cone, the slabs of water are disks, with thickness Δy. Since the right-hand side of the tank is a line in the plane with equation $y = -x/2 + 10$, the radius of an arbitrary disk of water at height y is $20 - 2y$. Hence, the volume of an arbitrary disk of water is $\pi(20 - 2y)^2\Delta y$. Hence, the force exerted by this disk of water downward is $\delta \cdot \pi(20 - 2y)^2\Delta y$. Since the distance from the top of the tank to a disk of water at height y is $10 - y$, the work required to pump this disk of water out of the tank is $(10 - y) \cdot \delta \cdot \pi(20 - 2y)^2 \Delta y$.

Since the tank is full to the brim with water, the total amount of work required to pump all the water out of the tank is $\int_0^{10} (10 - y) \cdot \delta \cdot \pi(20 - 2y)^2 \, dy$. This computation is trivial; we omit it here. The total amount of work is $624,000\pi$ ft$\cdot$ lb. □

5. Find the centroid of the region bounded by the curve $y = \cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi/2$.

Solution. Since $y = \cos x$ is even (hence symmetric about the y-axis) and the interval given ($[\pi/2, \pi/2]$) is symmetric, the x-coordinate of the centroid is 0.

To compute the y-coordinate $\overline{y}$ of the centroid, recall the formula for the y-coordinate of the centroid; we use

$$\overline{y} = \frac{1/2 \int_{-\pi/2}^{\pi/2} \cos^2 x \, dx}{\int_{-\pi/2}^{\pi/2} \cos x \, dx}$$

The denominator of $\overline{y}$ is easy to compute; the value is 2. For the numerator, we use the half-angle formula $\cos^2 x = \frac{1 + \cos 2x}{2}$. Then

$$\begin{aligned} 1/2 \int_{-\pi/2}^{\pi/2} \cos^2 x \, dx &= 1/2 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2x}{2} \, dx \\ &= 1/4 \Big|_{-\pi/2}^{\pi/2} x + \frac{1}{2} \sin 2x \\ &= \pi/4. \end{aligned}$$

Hence, the y-coordinate of the centroid is $\pi/8$, so the centroid is the point $(0, \pi/8)$. □