$$L(y) = 1 + \int_{\max\{target-A,\, y-B\}}^{\min\{target+A,\, y+B\}} L(x) f(x) dx$$

$$= 1 + \int_{target-A}^{target+A} I[y-B < x < y+B] \cdot L(x) \cdot f(x) dx$$

where $I[y-B < x < y+B]$

$$= \begin{cases} 1 & \text{if } y-B < x < y+B \\ 0 & \text{otherwise} \end{cases}$$

By using a quadrature rule for integrals on $[target-A,\, target+A]$ I can hopefully get a numerical version of $L(y)$ on interval.

$$\text{ARL} = 1 \cdot P[|x_1 - target| > A] + \int_{target-A}^{target+A} (1 + L(y)) f(y) dy$$

$$= 1 + \int_{target-A}^{target+A} L(y) f(y) dy.$$

I have (from the solution for $L(y)$) values $L(a_i)$ and above. I then replace the integral with

$$\sum w_i L(a_i) f(a_i)$$

and thereby find ARLs.

Crowder has done this for normal x — see his X/MR program on Professor Vardeman's Stat 531 Web page or tables in V&J — be warned that this program is somewhat unstable.

Tables 4.16 (p. 172), A.6 (p. 518, 519)

Some facts gotten from looking at X/MR ARLs:

1) "Standard" ("3 sigma") limits for X/MR combinations produce a very small "all OK" ARL.

2) Fairly extensive ARL studies have suggested that X alone beats X/MR (for a given all-OK ARL one gets much better ARLs for shifts in μ

and nearly as good ARLs for increases in σ using X alone)

3) The one type of non-all-OK behavior that MR helps detect is oscillation (e.g., oscillating mean for independent observations or time series behavior with negative autocorrelation).

Section 3.6 of V&J and Ch. 3 of the Notes

<u>Engineering/Feedback Control</u> (different from SPC, Statistical Process Control)

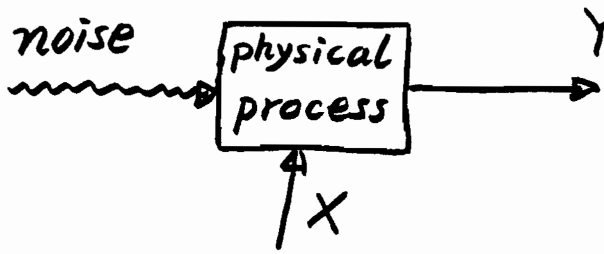
SPC/Process Monitoring	Engineering/Feedback Control
process <u>watching</u> + unspecified physical intervention	automated process tweaking/knob turning
detection methodology	Create stability

Here I'll discuss/outline

V&J → 1) discrete PID control
notes → 2) a simple version of "minimum variance" stochastic control theory

---

PID Control (in discrete time)



It is Y that we wish to "control" or guide to some target by means of changing X.

The point is usually not detailed modeling of how X impacts Y ... instead practice is more empirical based on information as crude as "Y tends to increase in X."

The notion is that for a sequence of targets

$$\ldots, T(1), T(2), T(3), \cdots$$

and observations

$$\ldots, Y(1), Y(2), Y(3), \cdots$$

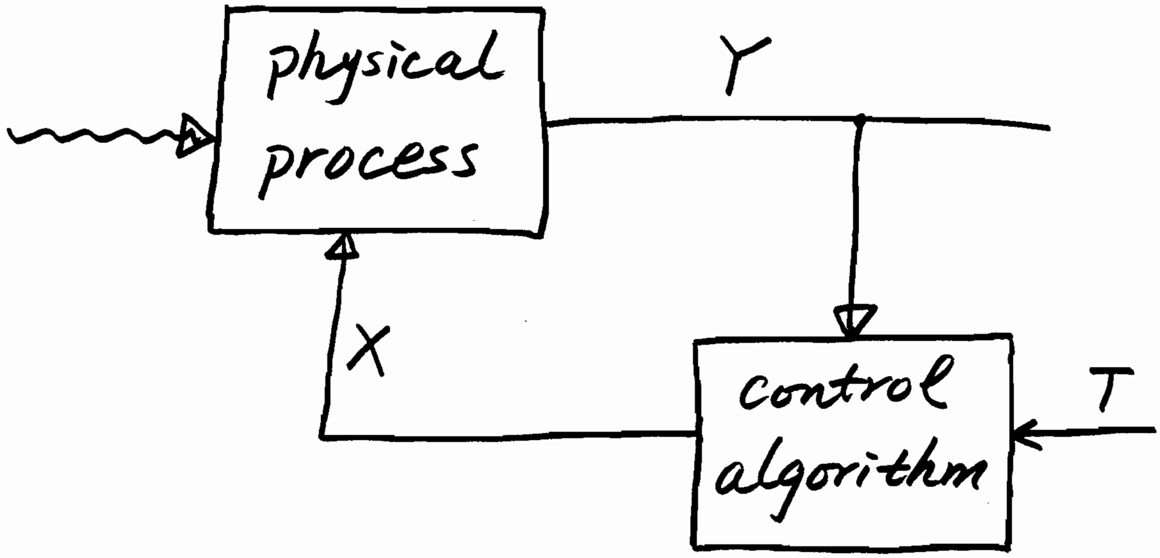
there are corresponding errors

$$E(t) = T(t) - Y(t)$$

and based on these one chooses adjustments to the manipulated variable (X)

$$\Delta X(t) = X(t) - X(t-1)$$

hoping to make E's small.



With

$$\Delta W(t) = W(t) - W(t-1)$$

$$\begin{aligned}\Delta^2 W(t) &= \Delta(\Delta W(t)) \\ &= \Delta\left(W(t) - W(t-1)\right) \\ &= W(t) - W(t-1) - (W(t-1) \\ &\quad - W(t-2)) \\ &= W(t) - 2W(t-1) + W(t-2),\end{aligned}$$

a "PID" control algorithm is

$$\Delta X(t) = \underbrace{K_1 \Delta E(t)}_{\text{The "proportional" part of the algorithm}} + \underbrace{K_2 E(t)}_{\text{The "integral" part of the algorithm}} + \underbrace{K_3 \Delta^2 E(t)}_{\text{The "derivative" part of the algorithm}}$$

K_1, K_2, K_3 control gains

<u>Note</u>: $$\Delta X(t) = (K_1 + K_2 + K_3) E(t) - (K_1 + 2K_3) E(t-1) + K_3 E(t-2)$$

Assume Y tends to increase in X.

"Integral-only control" $K_2 > 0,\ K_1 = K_3 = 0$

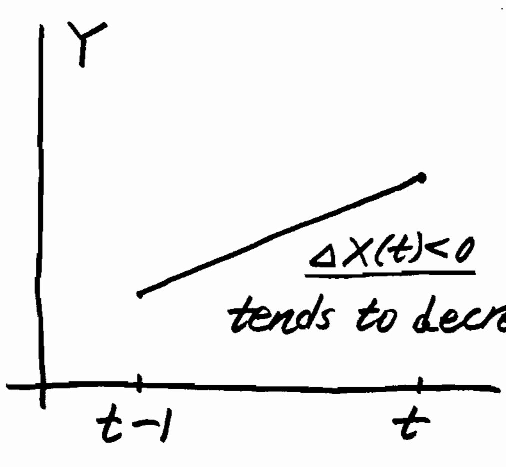
$$\Delta X(t) = K_2 E(t) = K_2 \left(T(t) - Y(t) \right)$$

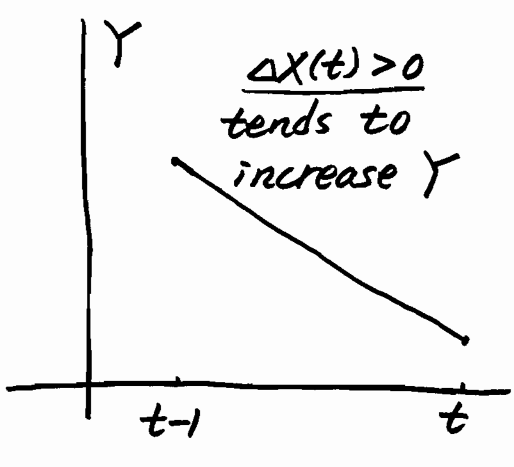
If $E(t) > 0$ then $T(t) > Y(t)$ and $\Delta X(t) > 0$ and I expect to increase Y;
if $E(t) < 0$, then $\Delta X(t) < 0$ and I expect to decrease Y.
This tends to put Y on-target, i.e., correct for mis-aim or offset.

"Proportional-only control" $K_1 > 0,\ K_2 = K_3 = 0.$

$$\Delta X(t) = K_1 \Delta E(t) = -K_1 \Delta Y(t)$$

constant T situation





So the proportional part of a PID algorithm tends to produce 0 slope, that is, keep E's (Y's for constant T) constant.

"Derivative-only control" $K_3 > 0$ and $K_1 = K_2 = 0$

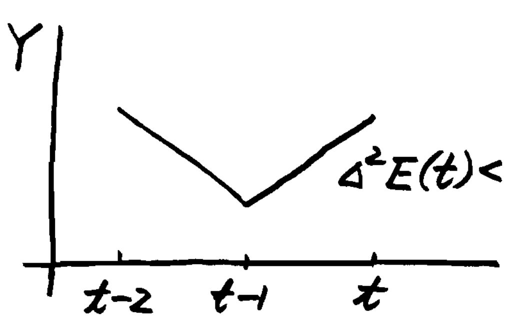
$$\Delta X(t) = K_3 \Delta^2 E(t)$$

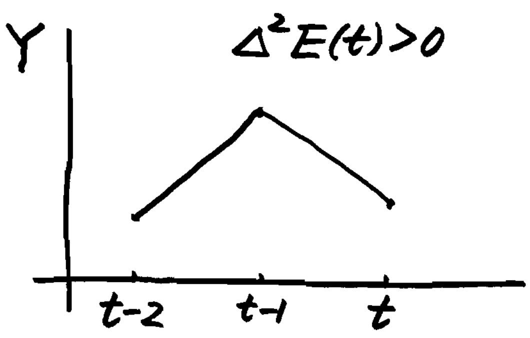
constant T

$$= -K_3 \Delta^2 Y(t)$$

$$= -K_3 \left(Y(t) - 2Y(t-1) + Y(t-2) \right)$$

(Constant T pictures)

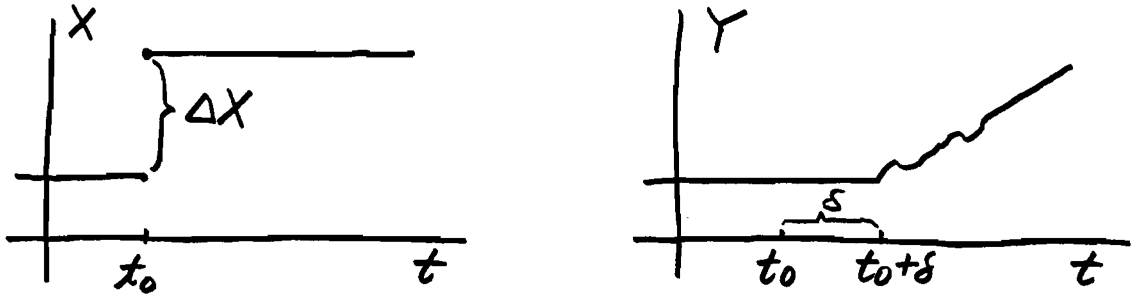




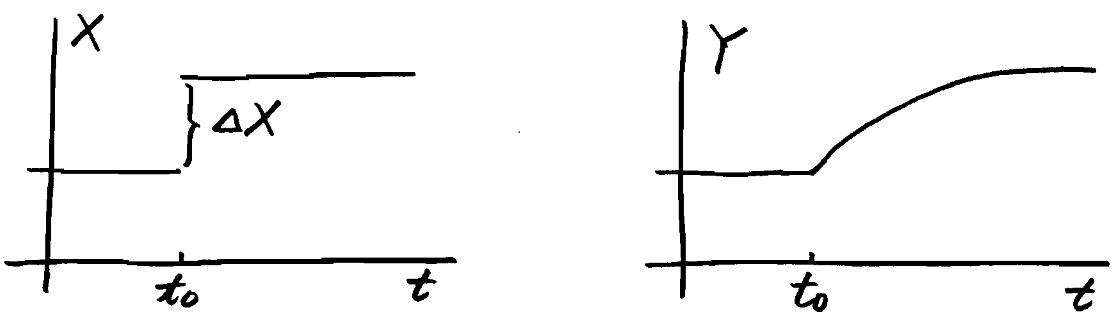
$\Delta X(t) = K_3 \Delta^2 E(t)$ will work to kill off non-zero curvature in a plot of $E(t)$ (or $Y(t)$ for constant T case)

Some facts of life that make control / on-line adjustment tricky:

1) There can be time delays, i.e., the effect of a change in X may not even begin to be seen until some time after the change is made.



2) Sometimes the effect of a change in X at time t_0 is only realized "gradually."



The big issue in applications of PID control is "tuning" or the choice of K_1, K_2, K_3.

Practical choice of these is an empirical problem. This can be approached as a (response surface) experimental design problem where I try to optimize process performance by choice of K_1, K_2, K_3, e.g., one might use an empirical measure of performance

$$S = \frac{1}{m} \sum_{t=1}^{m} (E(t))^2$$

for m large enough to wash out transient or start-up effects.

A theoretical analysis aimed at guiding choice of a control algorithm (e.g., K_1, K_2, K_3 in the case of PID) must have as ingredients

- some kind of measure to be optimized
- model for uncontrolled process behavior (deterministic or stochastic)
- model for how control actions will impact process output

Chapter 3 of the Notes a little bit of theory of optimal stochastic control

$$\ldots, Z(-1), Z(0), Z(1), \ldots$$

uncontrolled process

$\mathcal{F}$ is a probability model for $\{Z(t)\}$

e.g., 1) $Z(t) = Z(t-1) + \varepsilon(t)$

2) $Z(t) = Z(t-1) + d + \varepsilon(t)$

3) some Box-Jenkins ARIMA type model.

e.g., $Z(t) = \alpha Z(t-1) + \varepsilon(t)$

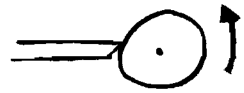
or $Z(t) = \varepsilon(t-1) + \varepsilon(t)$

Notation:

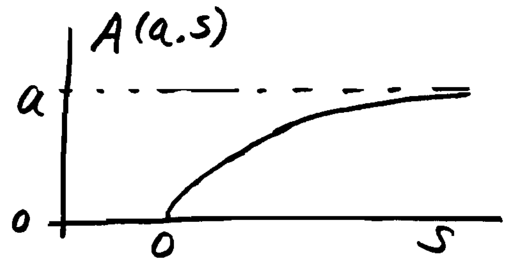
$a(t)$ control action taken at time t

$A(a,s)$ current impact of an action taken s periods ago

e.g., 1) $A(a,s) = a$ for $s \geq 1$ in a machine tool problem



2) $A(a,s) = a\left(1 - \exp\left(-\frac{sh}{T}\right)\right)$ for $s \geq 1$



3) $A(a,s) = a \exp\left(-\frac{sh}{T}\right)$ for $s \geq 1$

etc.

Observed (controlled) process behavior

$$Y(t) = Z(t) + \sum_{s=0}^{t-1} A(a(s), t-s)$$

and for $t \geq 0$ $a(t)$ will be based on

$$\ldots, Z(-1), Z(0), Y(1), Y(2), \ldots, Y(t)$$

a common objective is to minimize

$$E_{\mathcal{F}}(Y(t) - T(t))^2$$

(or the sum thereof)

Minimum Variance (MV) Control

Solution?

Note 1st that from

$$\ldots, Z(-1), Z(0), Y(1), Y(2), \cdots, Y(t)$$

I can recover

$$\ldots, Z(-1), Z(0), Z(1), Z(2), \cdots, Z(t)$$

$$Z(s) = Y(s) - \sum_{r=0}^{s-1} A(a(r), s-r)$$

Note also that at least in theory $\mathcal{F}$ gives conditional dsn for

$$Z(t+1), Z(t+2), Z(t+3), \cdots \qquad (*)$$

given

$$\ldots, Z(-1), Z(0), Z(1), \cdots, Z(t) \qquad (**)$$

and thus means for $(*)$ given $(**)$ and thus conditional means for the future given

$$\ldots, Z(-1), Z(0), Y(1), \cdots, Y(t)$$

(since I can recover Z's from Y's)

Let

$$E_{\mathcal{F}}[z(s) \mid \ldots, z(-1), z(0), \ldots, z(t)]$$

$$= E_{\mathcal{F}}[z(s) \mid z^t] = \hat{z}(s|t).$$

Suppose there are $u \geq 0$ periods of dead time (u could be 0).

— The earliest Y I can affect by $a(t)$ is $Y(t+1+u)$.

Then a natural projection of $Y(t+1+u)$ is

$$Y(t+1+u|t) \doteq \hat{z}(t+1+u|t) + \sum_{s=0}^{t-1} A(a(s), t+1+u-s) + A(a(t), u+1).$$

If I want $Y(t+1+u)$ "$\doteq$" $T(t+1+u)$

I should therefore try to choose $a(t)$ so that

$$A(a(t), u+1) = T(t+1+u) - \left\{ \hat{z}((t+1+u) \mid t) + \sum_{s=0}^{t-1} A(a(s), t+1+u-s) \right\}$$

This is <u>MV</u> control algorithm.
(Minimum Variance)