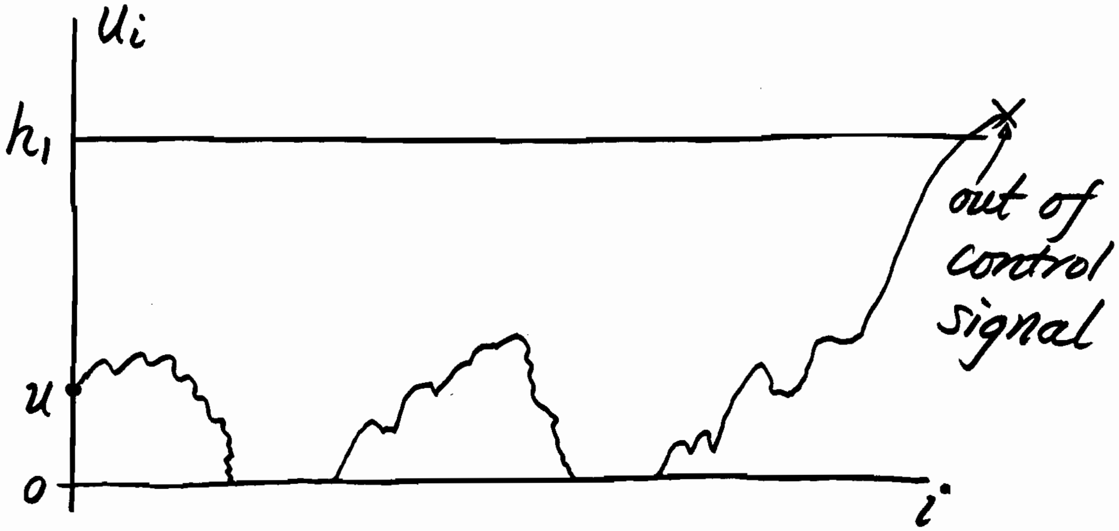
High Side "Decision Interval" CUSUM Scheme

For a reference value k_1, some starting value $U_0 = u \geq 0$ (sometimes called the "headstart") and some "decision interval" $h_1 > 0$, let

$$U_i = \max\left[0,\ U_{i-1} + (Q_i - k_1)\right]$$

and signal the first time U_i exceeds h_1.

— This is a CUSUM with a "restart" feature (to be used if it falls negative) that signals when it exceeds h_1.



— This is meant to catch the eventuality of "large"/"increased" μ_Q.

Low-Side "Decision Interval" CUSUM Scheme

For a reference value k_2, some starting value $V_0 = v_0 \leq 0$ and a decision interval $h_2 \geq 0$, let

$$V_i = \min[0, V_{i-1} + (Q_i - k_2)]$$

and signal when V_i first falls below $-h_2$.

- This is meant to catch the eventuality of "small"/"decreased" μ_Q.

Combined schemes (simultaneous high- & low-side schemes) can be used to monitor for <u>any</u> change in μ_Q.

- Choice of parameters $(k_1, h_1, k_2, h_2, u, v)$: see §4.2 of V&J
- ARL behavior of scheme (for a given set of parameters and model for Q sequence): It turns out that analysis of 1-sided scheme behavior is usually sufficient. (<u>Tools</u>: 1) MC's. 2) numerical solutions to integral equations)

Fact (Yashchin) $Q_1, Q_2, \cdots$, iid. If $L_1(u)$ and $L_2(v)$ are respective ARL's of separate high and low side decision interval schemes with starting values u and v; $k_1 \geq k_2$, and $(k_1 - k_2) - |h_1 - h_2| \geq \max[0, u - v - \max(h_1, h_2)]$.

then

$$ARL_{comb.} = \frac{L_1(0)\cdot L_2(v) + L_2(0)\cdot L_1(u) - L_1(0)\cdot L_2(0)}{L_1(0) + L_2(0)}.$$

Note: in the common case where $h_1 = h_2 = h$ and $u = v = 0$ this simplifies.

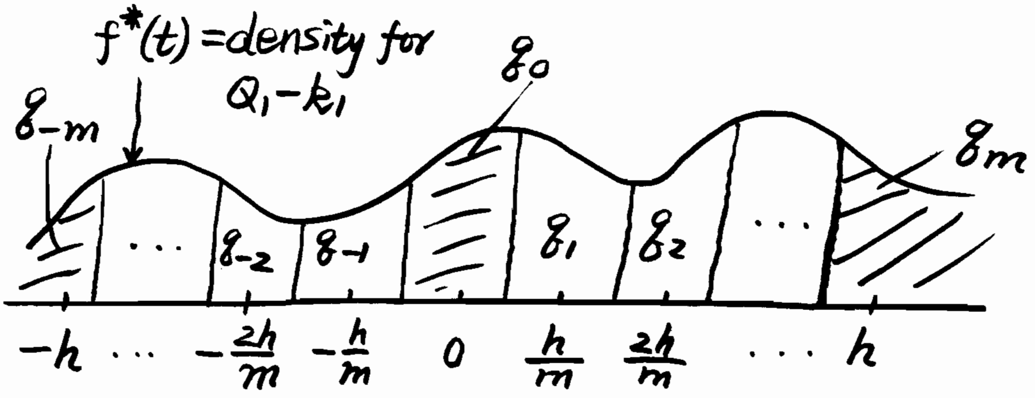
MC Methods for CUSUM ARL's

(for cont$\underline{s}$ Q via discretization)

Abbreviate h_1 to h (at least temporarily). Let

$Q_i^* = Q_i - k_1$ rounded to nearest multiple of $\frac{h}{m}$, where m is some suitably large integer.

To get approximate ARL's for a high-side scheme based on Q_i's consider a high side scheme based on Q_i^*'s (with 0 reference value).



Use a MC with states

S_i = "no alarm yet and current CUSUM (based on rounded values) is $i \cdot (\frac{h}{m})$"

for $i = 0, 1, 2, 3, \cdots, m-1$ and

S_m = "alarm".

If I can write down an appropriate P then L_i is an approximate ARL using headstart $i(\frac{h}{m})$, i.e.,

$(I-R)^{-1}1 = L$ ← entries are ARL's:

$L_0, L_1, L_2, \cdots, L_{m-1}$

$L_i \simeq$ ARL from $i(\frac{h}{m})$ headstart.

$P = ?$ (see next page)

$$\begin{array}{c|ccccc|c}
 & S_0 & S_1 & S_2 & \cdots & S_{m-1} & S_m=\text{alarm} \\ \cline{2-6}
S_0 & \sum_{j=-m}^{0} q_j & q_1 & q_2 & \cdots & q_{m-1} & q_m \\
S_1 & \sum_{j=-m}^{-1} q_j & q_0 & q_1 & \cdots & q_{m-2} & q_{m-1}+q_m \\
S_2 & \sum_{j=-m}^{-2} q_j & q_{-1} & q_0 & \cdots & q_{m-3} & q_{m-2}+q_{m-1}+q_m \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
S_{m-1} & q_{-m}+q_{-m+1} & q_{-m+2} & q_{-m+3} & \cdots & q_0 & \sum_{j=1}^{m} q_j \\ \cline{2-6}
\underset{\overset{\uparrow}{\text{alarm}}}{S_m} & 0 & 0 & 0 & \cdots & 0 & 1 \\
 & & & & \overset{\uparrow}{R} & &
\end{array}$$

"The most common dsns for Q_i used to make such ARL computations are normal — based on normal ARL calculations "standard" means of choosing parameters have been developed.

Numerical Solutions of Integral Equations to Produce ARL's.

Here we will again consider iid Q with marginal density f.

$L_1(u)$ = ARL for a high side scheme with headstart u (and decision interval h_1 and reference value k_1)

$$L_1(u) = 1 \cdot P[Q_1 - k_1 \geq h_1 - u] + (1 + L_1(0)) \cdot P[Q_1 - k_1 \leq -u] + \int_{k_1 - u}^{k_1 + h_1 - u} (1 + L_1(u + t - k_1)) f(t)\,dt$$

$$\boxed{L_1(u) = 1 + L_1(0) F(k_1 - u) + \int_0^{h_1} L_1(t) f(t + k_1 - u)\,dt}$$

Solutions?

Usually one must resort to numerical methods... This involves approximating the integral.

Suppose I wish to approximate integrals on $[a, a+h]$ for "reasonable" functions g (i.e., I'm interested in $\int_a^{a+h} g(t)\,dt$).

I can choose a quadrature method consisting of points

$$a \le a_1 \le a_2 \le a_3 \le \cdots \le a_m \le a+h$$

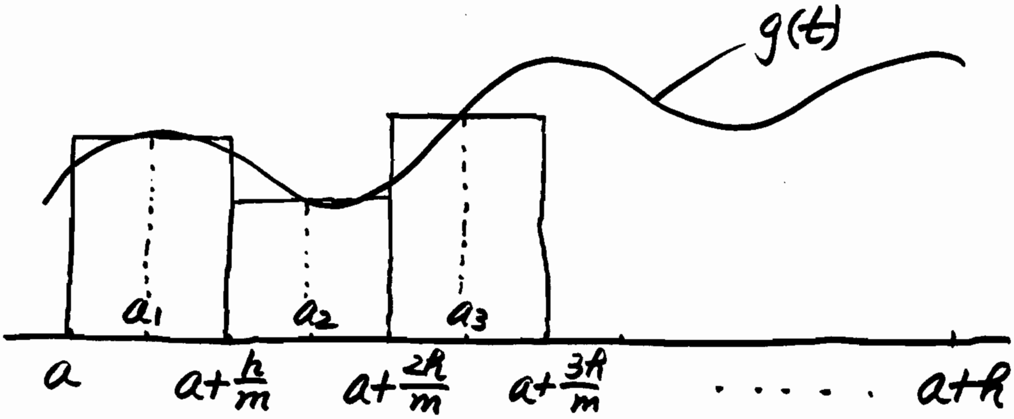
and weights

$$w_i \ge 0 \quad \text{with } \sum w_i = h$$

and approximate

$$\int_a^{a+h} g(t)\,dt \approx \sum_{i=1}^{m} w_i\, g(a_i).$$

E.g., $a_i = a + \frac{i-.5}{m} h$ and $w_i = \frac{h}{m}$ gives the simple/crude "histogram" approximation to the integral.



Given an m point quadrature rule for $[0, h_1]$ I can write

$$L_1(u) \approx 1 + L_1(a_1)F(k_1 - u) + \sum_{j=1}^{m} w_j L_1(a_j) f(a_j + k_1 - u)$$

$$(L_1(0) \approx L_1(a_1))$$

If I knew $L_1(a_1), L_1(a_2), \cdots, L_1(a_m)$ I would have an approximate form for the function $L_1(u)$.

<u>Idea</u>: Solve for $L_1(a_1), L_1(a_2), \cdots, L_1(a_m)$

Use the above expression at $u = a_1, a_2, \cdots, a_m$ to get a set of linear equations for $L_1(a_1), L_1(a_2), \cdots, L_1(a_m)$:

$$\left[\begin{aligned} L_1(a_i) &\approx 1 + L_1(a_1)F(k_1 - a_i) \\ &\quad + \sum_{j=1}^{m} w_j L_1(a_j) \cdot f(a_j + k_1 - a_i) \end{aligned}\right.$$

m equations

It turns out that one can write this set of m equations in the form

$$L = 1 + RL$$

where now R is not quite the MC R matrix but is very close; see development on pages 31—33 of the notes for L, R.

Note that entries in columns 2 through m of this R look like those in MC analysis. .... Why?

Suppose I'm using

$$a_i = \left(\frac{i-.5}{m}\right) \cdot h_1 \qquad w_i = \frac{h_1}{m}.$$

Then a generic entry (row i column j) here is

$$w_j f(a_j - a_i + k_1) = \frac{h_1}{m} f\left(\frac{h_1}{m}(j-i) + k_1\right)$$

$$= \frac{h_1}{m} f^*\left(\frac{h_1}{m}(j-i)\right)$$

$$\approx \int_{(j-i)\frac{h_1}{m} - \frac{h_1}{2m}}^{(j-i)\frac{h_1}{m} + \frac{h_1}{2m}} f^*(t)\,dt$$

$$= g_{j-i}.$$

So the two approaches are nearly the same at the end of the day.

One benefit derived from the present analysis is a nice interpolation formula

$$L_1(u) \approx 1 + L_1(a_1) F(k_1 - u) + \sum_{j=1}^{m} w_j L_1(a_j) f(a_j + k_1 - u)$$

"Obviously" the normal Q version of this is important.

Note that the following 3 CUSUMs (high side) are identical in behavior:

Values CUSUMed	Reference value	Decision interval
Q_i	k_1	h_1
Q_i/σ_Q	k_1/σ_Q	h_1/σ_Q
$(Q_i - \mu_Q)/\sigma_Q$	$(k_1 - \mu_Q)/\sigma_Q$	h_1/σ_Q

So "clearly"

$N(\mu_Q, \sigma_Q^2)$ properties of CUSUM of
Q_i with reference value k_1 and
decision interval h_1

||

$N(0,1)$ properties of a scheme with
reference value $= \frac{k_1 - \mu_Q}{\sigma_Q} = -\mathcal{K}^*$

("standardized difference from reference value") (see (4.16), (4.17) of V&J)

and
decision interval $= \frac{h_1}{\sigma_Q} = \mathcal{H}^*$ (see (4.15) of V&J)

Note that for $N(0,1)$ case

1) Gan's program provides CUSUM ARL's (and quantiles of the run length dsn) for one-sided schemes

2) Table A.4 (page 515) of V&J gives 0-headstart ARL's (from Gan)

3) Siegmund's approximation provides a somewhat crude but explicit formula for approximate ARL's (one-sided, 0-headstart).

In the notation of (4.15) — (4.17) of V&J: ARL $\approx$

$$\begin{cases} (\mathcal{H}^* + 1.166)^2 & \text{if } \varphi^* = 0 \\ \dfrac{\exp[-2\varphi^*(\mathcal{H}^* + 1.166)] - 1 + 2\varphi^*(\mathcal{H}^* + 1.166)}{2(\varphi^*)^2} & \text{if } \varphi^* \neq 0 \end{cases}$$

Siegmund's ARL Approximation for Normal One-Sided CUSUMs

In the notations (4.15)-(4.17) of Vardeman & Jobe, Siegmund's approximation for a one-sided (0 head start) CUSUM ARL (for normal Q) is

$$ARL \approx \begin{cases} (\mathcal{H}^* + 1.166)^2 & \text{if } \mathcal{S}^* = 0 \\ \dfrac{\exp(-2\mathcal{S}^*(\mathcal{H}^* + 1.166)) + 2\mathcal{S}^*(\mathcal{H}^* + 1.166) - 1}{2(\mathcal{S}^*)^2} & \text{if } \mathcal{S}^* \neq 0 \end{cases} .$$

(Values from this approximation are to be compared to entries in Table A.4 of V&J.)