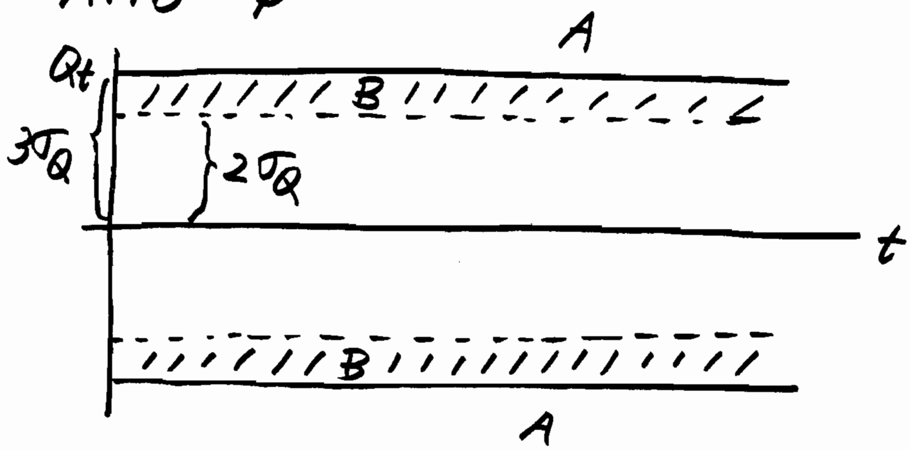
Example (Wetherill)

"Shewhart" Chart with 1 extra alarm rule:

"one point in region A"

"2 points in a row in region B"

$A \cap B = \phi$



Suppose $Q_1, Q_2, \cdots$ iid with some (fixed marginal) dsn (it need not be the one used to define A and B)

ARL = ?

Let $q_A = P[Q_1 \text{ is in region } A]$

$q_B = P[Q_1 \text{ is in region } B]$

$L = ARL$

L^* = mean # of additional points needed to produce a signal if there has been none to date and present Q is in B.

Then $L = 1 \cdot q_A + (1+L^*) \cdot q_B + (1+L) \cdot (1-q_A-q_B)$

$$= 1 + L \cdot (1-q_A-q_B) + L^* \cdot q_B$$

$$L^* = 1 \cdot (q_A+q_B) + (1+L) \cdot (1-q_A-q_B)$$

$$= 1 + L(1-q_A-q_B).$$

These are 2 <u>linear</u> equations in L and L^*. That can be solved simultaneously and

$$L = \frac{1+q_B}{1-(1-q_A-q_B)-q_B(1-q_A-q_B)}$$

<u>Aside</u>: An application of this (exactly as just developed) might be to the following

$Q = X =$ a count

Invent a scheme "signal the first time the sum of 2 consecutive X's is ≥ 2."

This is exactly like the Wetherill example with

$$q_B = f(1) = P[X=1]$$

$$q_A = 1-f(0)-f(1) = P[X \geq 2]$$

(assuming X's are iid)

Note also that I can even generalize the Wetherill example slightly and have a non-iid model for Q's —

e.g. let

$$q_A = P[Q_1 \text{ is in region } A]$$
$$= P[Q_{t+1} \text{ is in region } A / Q_t \text{ is in } (A \cup B)^c \text{ (and any sequence before } Q_t)]$$

$$q_B = P[Q_1 \text{ is in region } B]$$
$$= P[Q_{t+1} \text{ is in } B / Q_t \text{ is in } (A \cup B)^c]$$

and

$$q_A^* = P[Q_{t+1} \text{ is in } A / Q_t \text{ is in } B]$$
$$q_B^* = P[Q_{t+1} \text{ is in } B \mid Q_t \text{ is in } B]$$

(Here I no longer require $q_A = q_A^*$ or $q_B = q_B^*$.)

This is now a non-iid model for Q sequence. — With L and L^* as before I can still write linear equations for them:

$$L = 1 \cdot q_A + (1+L^*) \cdot q_B + (1+L) \cdot (1 - q_A - q_B)$$
$$L^* = 1 \cdot (q_A^* + q_B^*) + (1+L) \cdot (1 - q_A^* - q_B^*).$$

Application of Discrete Time / Discrete State Space Markov Chains to Analysis of Process Monitoring Schemes

Markov Chains — models for systems that at discrete times $1, 2, 3, \cdots$ bounce around between "states" $S_1, S_2, S_3, \cdots, S_m, S_{m+1}$.

(Markov Chains : MC's)

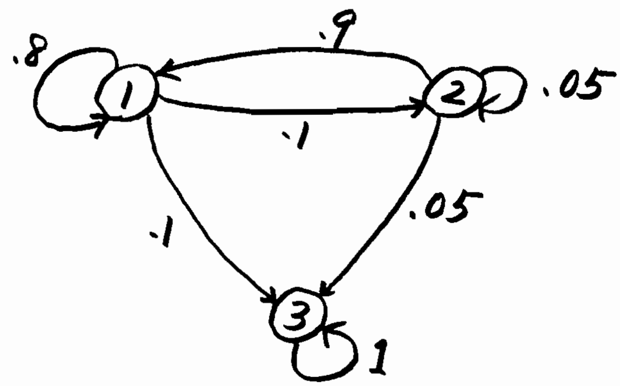
"Markov" assumption: conditional dsn of where I'll be at time $t+1$ given the entire history up through time t only depends upon the state at time t.

"Stationary" MC's employ the assumption that a single matrix of "transition probabilities" governs movement at all times

$$\underset{(m+1)\times(m+1)}{P} = (p_{ij})$$

$$p_{ij} = P[S_j \text{ at time } t+1 \mid S_i \text{ at time } t]$$

Example Simple 3-state system



$$\underset{3\times3}{P} = \begin{pmatrix} .8 & .1 & .1 \\ .9 & .05 & .05 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition: If $P_{ii}=1$, state i is called an absorbing state.

Of particular interest are MC's where S_{m+1} is absorbing (where it is possible at least eventually to move from any of $S_1, S_2, \cdots, S_m$ to S_{m+1}).

— These have associated matrix formulas for "mean times to absorption" that we can use to compute ARL's.

Let L_i = mean # of transitions required to move from S_i to S_{m+1}

$$\underset{m\times 1}{L} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix} \qquad \underset{(m+1)\times(m+1)}{P} = \begin{pmatrix} \underset{m\times m}{R} & \underset{m\times 1}{r} \\ \\ \underset{1\times m}{0} & \underset{1\times 1}{1} \end{pmatrix}$$

Then

$$\boxed{\underset{m\times 1}{L} = (\underset{\underset{m\times m}{\uparrow}}{I} - R)^{-1} \cdot \underset{m\times 1}{1}} \qquad (\underset{m\times m}{I} : \text{identity matrix})$$

Why?

$$L_1 = (1+L_1)P_{11} + (1+L_2)P_{12} + \cdots + (1+L_m)P_{1m} + 1\cdot P_{1,m+1}$$

$$L_2 = (1+L_1)P_{21} + (1+L_2)P_{22} + \cdots + (1+L_m)P_{2m} + 1\cdot P_{2,m+1}$$

$$\vdots$$

$$L_m = (1+L_1)P_{m1} + (1+L_2)P_{m2} + \cdots + (1+L_m)P_{mm} + 1\cdot P_{m,m+1}$$

$$L_1 = 1 + P_{11}L_1 + P_{12}L_2 + \cdots + P_{1m}L_m$$

$$L_2 = 1 + P_{21}L_1 + P_{22}L_2 + \cdots + P_{2m}L_m$$

$$\vdots$$

$$L_m = 1 + P_{m1}L_1 + P_{m2}L_2 + \cdots + P_{mm}L_m$$

That is, $\underset{m\times 1}{L} = \underset{m\times 1}{1} + \underset{m\times m}{R}\,\underset{m\times 1}{L}$

$$(I-R)L = 1$$

$$L = (I-R)^{-1}1$$

I−R is nonsingular in our context

<u>Example</u> $P = \begin{pmatrix} .8 & .1 & .1 \\ .9 & .05 & .05 \\ 0 & 0 & 1 \end{pmatrix}$ $R = \begin{pmatrix} .8 & .1 \\ .9 & .05 \end{pmatrix}$

and $(I-R)^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10.5 \\ 11 \end{pmatrix}$ $L_1 = 10.5$
$L_2 = 11$

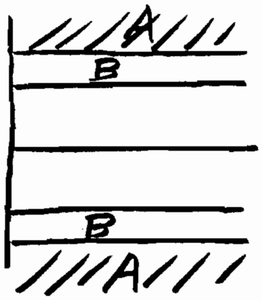
How to make use of this machinery?

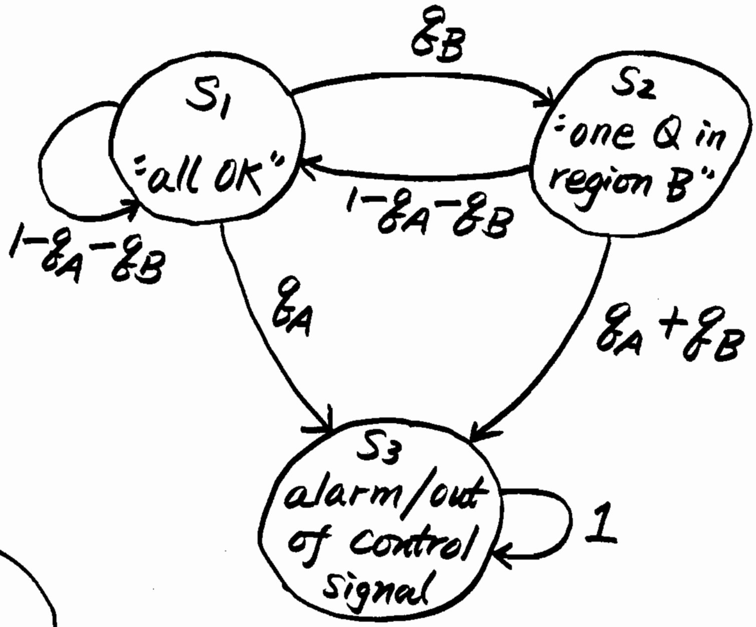
"Often" we can think up MC representation of monitoring scheme behavior with

S_{m+1} = "alarm"/out of control signal

and some element L becomes the ARL of interest.

Example Wetherill 2 alarm rule scheme





$L_1 = L = ARL$

$L_2 = L^*$

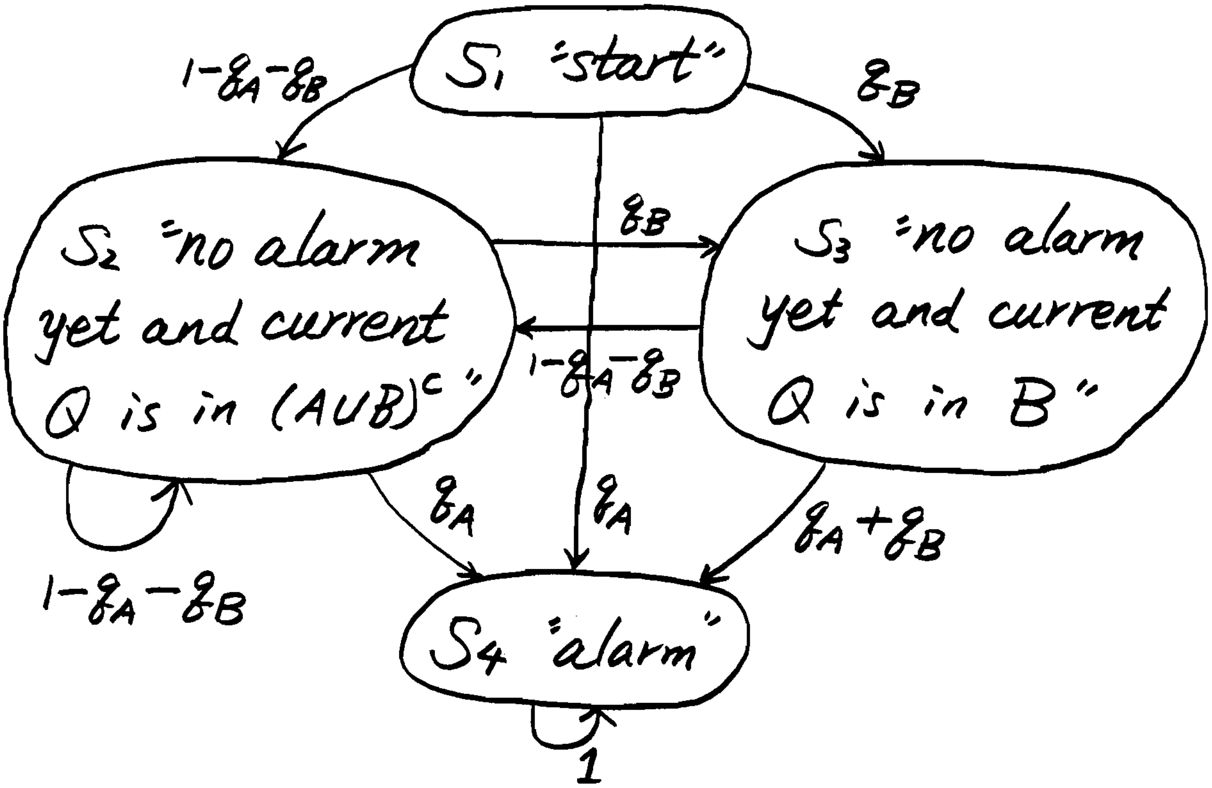
$(I-R)^{-1}\mathbf{1} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$

ARL of interest

See problem 2.22 of the notes for a matrix formula for 2nd moments of times to absorption, which can be used to get SDRL's.

There is no unique way of doing this, i.e., several possible representations may exist for a given scheme and model for Q's.

Example 2nd try at Wetherill scheme



$$P = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 0 & 1-q_A-q_B & q_B & q_A \\ 0 & 1-q_A-q_B & q_B & q_A \\ 0 & 1-q_A-q_B & 0 & q_A+q_B \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

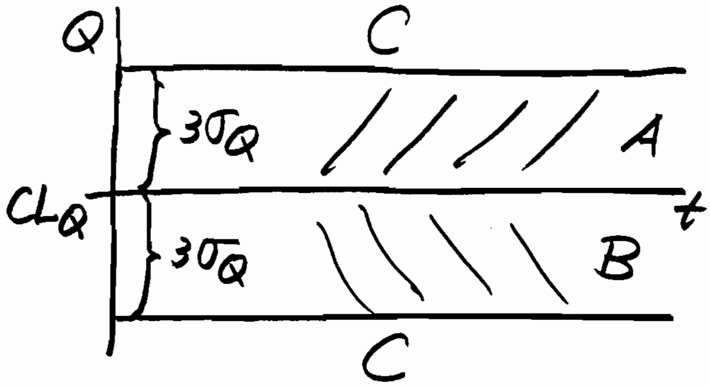
$\Rightarrow L_1 = L_2$ (because 1st 2 rows of P are the same)

and $ARL = L_1 = L_2$.

Example 2 Western Electric alarm rules

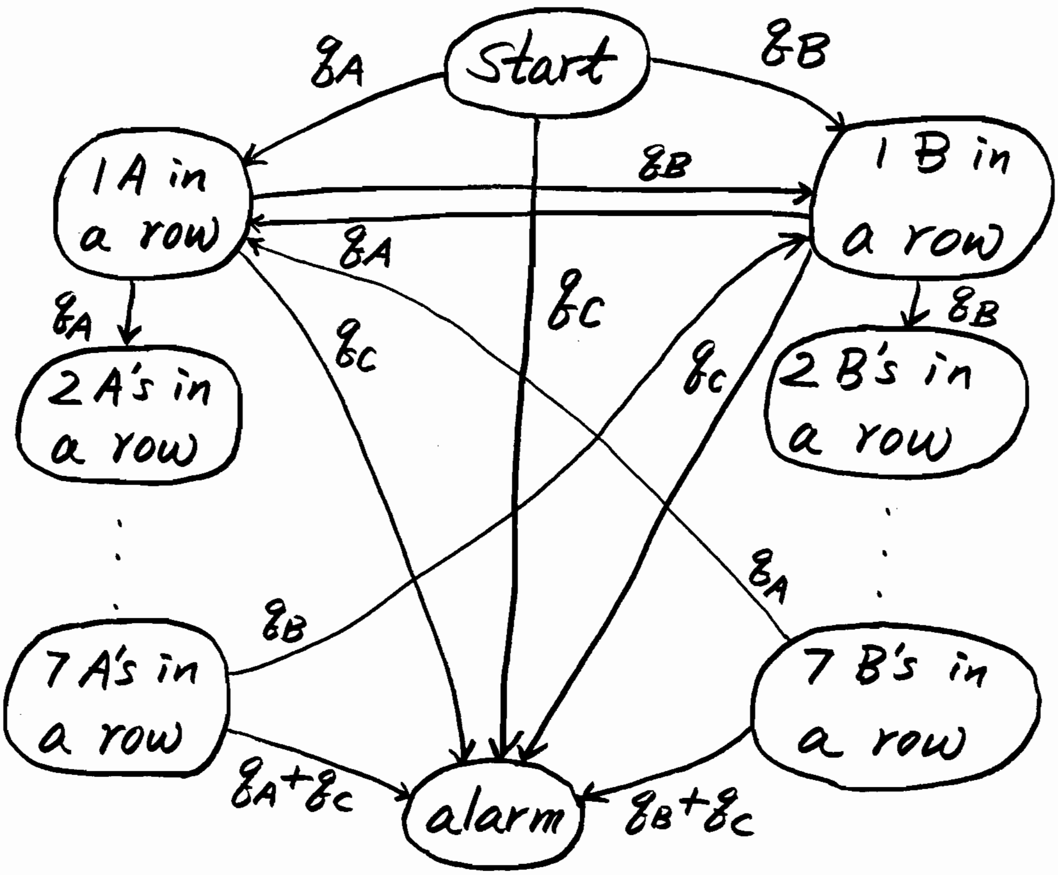
- a single point outside 3-sigma limits
- 8 consecutive points on a single side of center line

iid Q's ARL = ?



q_A, q_B, q_C

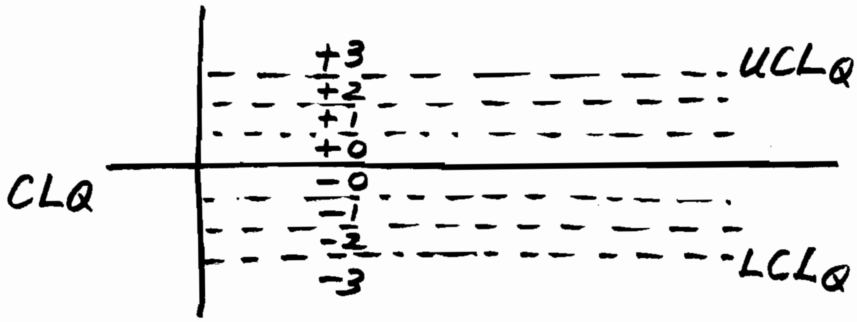
MC representation?



ARL = L_{start}

Example A so-called "Run-Sum" scheme—

This assigns scores to various zones on a Shewhart chart



Let Q_i^* = "score assigned to Q_i"

R_i = "total" of scores through period i where a new sum starts when

a score has a new sign

Alarm if $|Q_i^*| = 3$ or if $|R_i| \geq 4$.

E.g.

i	Q_i^*	R_i
1	+0	+0
2	+1	+1
3	+1	+2
4	+0	+2
5	−0	−0
6	−2	−2
7	−0	−2
8	−1	−3
9	+2	+2
10	+2	+4 ← alarm

ARL? Use MC ideas

For $i = \pm 0, \pm 1, \pm 2, \pm 3$, let

S_i = "no alarm yet and current run-sum is i"

Q iid with

$$q_j = P[Q^* = j]$$

for $j = \pm 0, \pm 1, \pm 2, \pm 3$.

So with a 9th state being an alarm state we get a big (9×9) transition matrix (recorded in the notes, page 26).

$$
\begin{array}{c|cccccccc|c}
R & S_{-0} & S_{-1} & S_{-2} & S_{-3} & S_{+0} & S_{+1} & S_{+2} & S_{+3} & \text{alarm} \\
\hline
S_{-0} & q_{-0} & q_{-1} & q_{-2} & 0 & q_{+0} & q_{+1} & q_{+2} & 0 & q_{-3}+q_{+3} \\
S_{-1} & 0 & q_{-0} & q_{-1} & q_{-2} & q_{+0} & q_{+1} & q_{+2} & 0 & q_{-3}+q_{+3} \\
S_{-2} & & & & & & & & & \\
S_{-3} & & & & & & & & & \\
S_{+0} & q_{-0} & q_{-1} & q_{-2} & 0 & q_{+0} & q_{+1} & q_{+2} & 0 & q_{-3}+q_{+3} \\
S_{+1} & & & & & & & & & \\
S_{+2} & & & & & & & & & \\
S_{+3} & & & & & & & & & \\
\hline
\text{alarm} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$(I-R)^{-1}\mathbf{1} = L \qquad L_{-0} = L_{+0} = ARL$

So ① We'll judge the "goodness" of a monitoring scheme on the basis of run length dsn properties

② MC's are tools for studying these

<u>Q</u>: What have people found out about which schemes have good ARL properties?/ What are the "best" available schemes?

<u>A</u>: CUSUM and EWMA schemes are among the best (see V&J 4.1 and 4.2)

<u>4.2 of V&J CUSUM monitoring schemes</u>

(and the analysis of their behavior)

<u>Basic Notation</u>: With values $Q_1, Q_2, \cdots$ and some "reference value" k and some starting value $CUSUM_0$ I can define

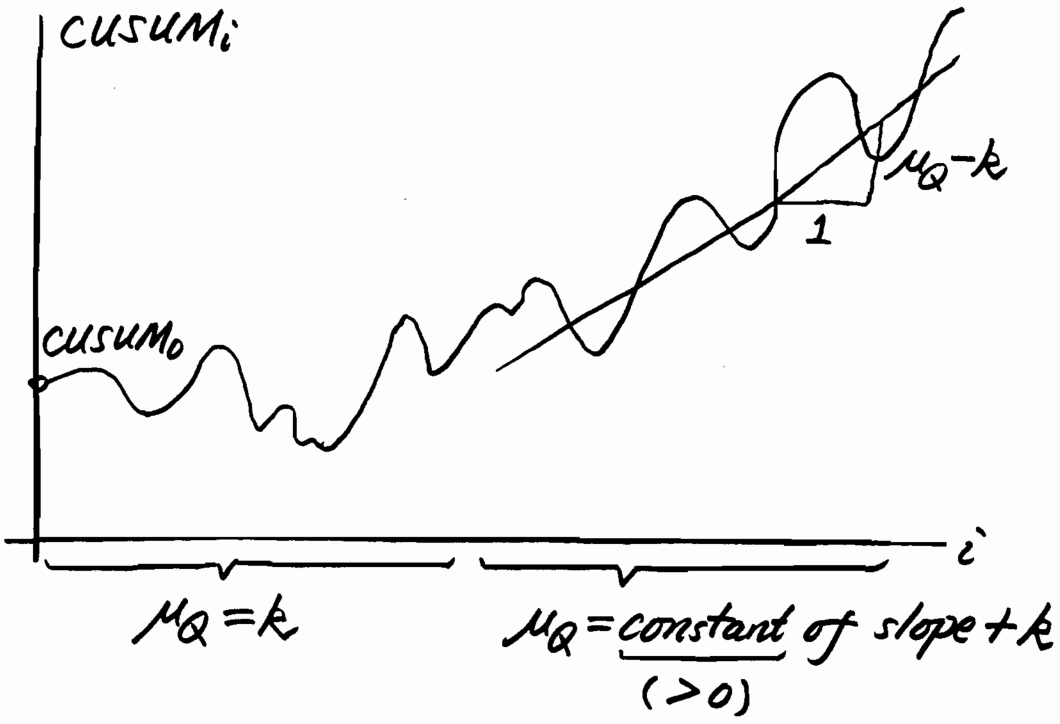
$$CUSUM_i = CUSUM_{i-1} + (Q_i - k)$$
$$\left(= CUSUM_0 + \sum_{j=1}^{i} (Q_j - k)\right)$$

and use this as basis for monitoring.

Note that if Q's are iid with mean μ_Q, how μ_Q compares with k will govern the behavior of the CUSUM sequence – CUSUM sequence will be a random walk with per period drift $\mu_Q - k$:

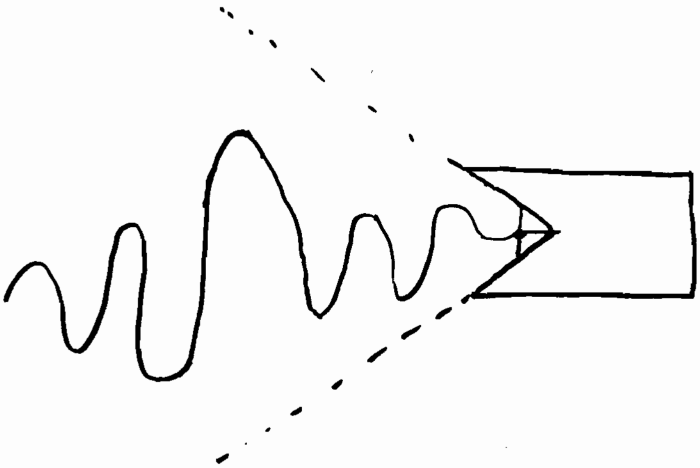
- If $\mu_Q - k = 0$ then CUSUM sequence will fluctuate about $CUSUM_0$.
- If $\mu_Q - k \neq 0$ then CUSUM sequence will drift up or down according to the sign of $\mu_Q - k$.

Hypothetical plot:



Alarm Rules?

One "possibility" is to use a V-mask.



Better: Consideration of High- and Low-Side Decision Interval CUSUM Schemes