How to use $L(\mu,\sigma)$?

I might make "maximum likelihood" (point) estimates of μ,σ by finding $(\hat{\mu},\hat{\sigma})$ that maximizes $L(\cdot,\cdot)$.

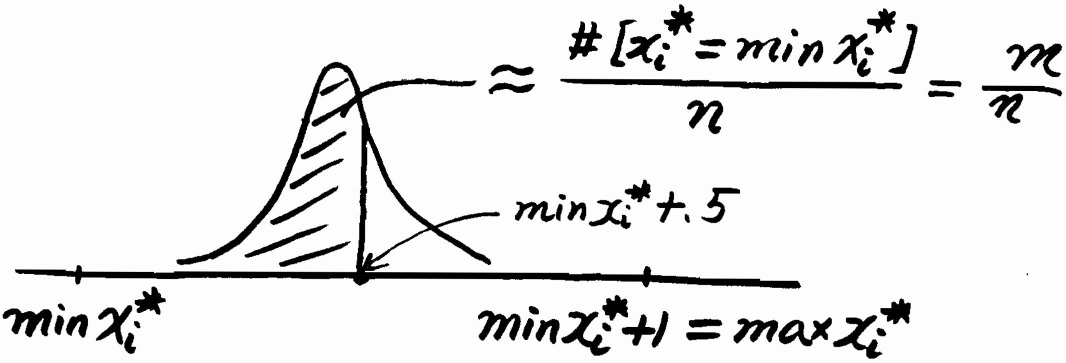
1. When the sample range of $x_1^*, x_2^*, \cdots, x_n^*$ is at least 2, $L(\mu,\sigma)$ is nice — it's mound-shaped and smooth and any sensible method will work to find maximizers $(\hat{\mu},\hat{\sigma})$.

(You can do this easily with Karen Jensen's CONEST program; see Professor Vardeman's Stat 531 Web page.)

2. When the sample range of $x_1^*, \cdots, x_n^*$ is 1 then all (μ,σ) pairs with σ small and

$$\Phi\left(\frac{\min x_i^* + .5 - \mu}{\sigma}\right) \approx \frac{\#[x_i^* = \min x_i^*]}{n}$$

have nearly the same (and nearly "maximum") likelihood $(L(\mu,\sigma))$.



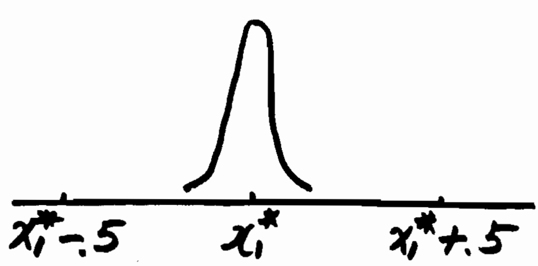
Here I will guess that σ is small and μ and σ are such that the normal (μ, σ^2) probability to the left of $\min x_i^* + .5$ is $\frac{\#[x_i^* = \min x_i^*]}{n}$. Let $m = \#[x_i^* = \min x_i^*]$. Then

$$\text{"}\max\text{"}\, L(\mu,\sigma) = \sup L(\mu,\sigma) = \left(\frac{m}{n}\right)^m \cdot \left(1 - \frac{m}{n}\right)^{n-m}.$$

3. When the sample range of $x_1^*, x_2^*, \cdots, x_n^*$ is 0 all (μ, σ) pairs with σ small and

$$x_1^* - .5 < \mu < x_1^* + .5$$

have nearly the same (and nearly "maximum") $L(\mu, \sigma)$.



Here I guess that σ is small and μ is within .5 of common x_i^*

$$\text{"}\max\text{"}\, L(\mu,\sigma) = \sup L(\mu,\sigma) = 1.$$

It is also possible to use $L(\mu,\sigma)$ to guide making of confidence sets for (μ,σ), or μ, or σ.

Confidence Sets for (μ,σ):

$$\text{Confidence Set} = \left\{ (\mu,\sigma) \;\middle|\; \begin{array}{l} L(\mu,\sigma) \text{ is not too much} \\ \text{smaller than "max" } L(\mu,\sigma) \end{array} \right\}$$

Large sample theory $\Rightarrow$ $(1-\alpha)$ confidence set for $(\mu,\sigma) = \left\{ (\mu,\sigma) \;\middle|\; \mathcal{L}(\mu,\sigma) > \sup_{\mu,\sigma} \mathcal{L}(\mu,\sigma) - \frac{1}{2}\chi_2^2 \right\}$

where

$$\mathcal{L}(\mu,\sigma) = \ln L(\mu,\sigma)$$

and χ_2^2 = upper α point of χ_2^2 dsn.

Confidence Intervals for μ:

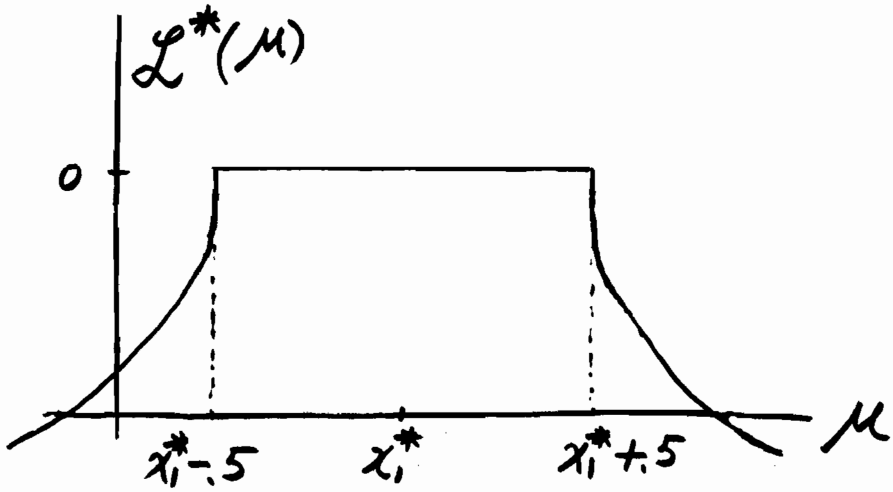
Let $L^*(\mu) = \sup_{\sigma} L(\mu,\sigma)$ "profile likelihood" for μ

Using the "most likely" value of σ for a given μ, this in some sense measures how likely μ is.

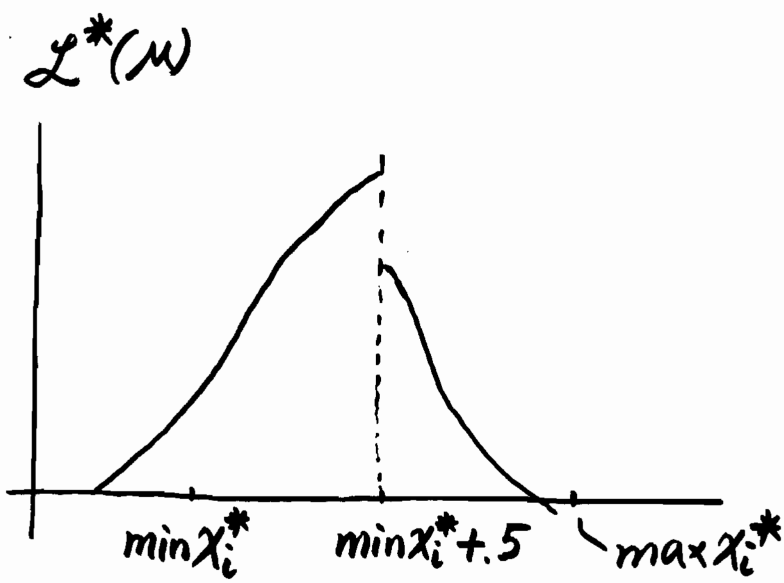
As it turns out there are 3 basic shapes for $L^*(\mu)$ depending upon range of $x_1^*, x_2^*, \cdots, x_n^*$.

Let $\mathcal{L}^*(\mu) = \ln L^*(\mu)$.

Range=0:

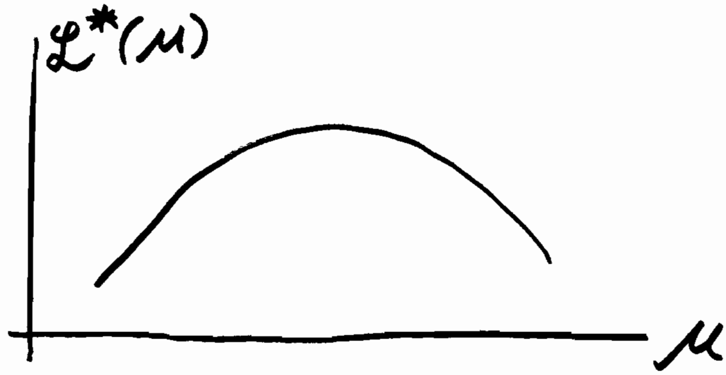


Range=1:



Case where there are more observations at $\min x_i^*$ than at $\max x_i^*$.

Range ≥ 2:

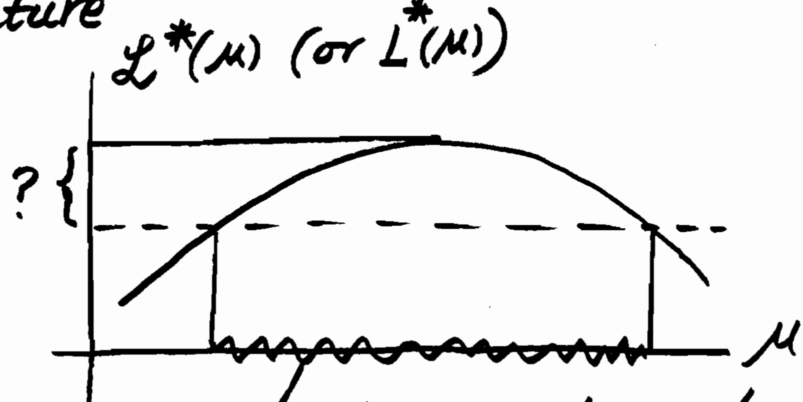


As a way of making CI's for μ

I'll use

$$\begin{array}{c}\text{confidence}\\ \text{interval}\end{array} = \left\{ \mu \;\middle|\; \begin{array}{l}\mathcal{L}^*(\mu) \text{ is not "too much"}\\ \text{below maximum possible}\end{array} \right\}$$

Picture



Large sample theory says that approximately

$$\begin{array}{c}(1-\alpha) \text{ level}\\ \text{CI for } \mu\end{array} = \left\{ \mu \;\middle|\; \mathcal{L}^*(\mu) > \sup_{\mu} \mathcal{L}^*(\mu) - \frac{1}{2}\chi_1^2 \right\}$$

upper α point of χ_1^2.

This doesn't work too well for small n.

- Happily Johnson Lee has found replacements for χ_1^2 that work fine for any n — $c(n,\alpha)$:

$$c(n,\alpha) = n \cdot \ln\left(\frac{t^2_{\alpha/2:(n-1)}}{n-1} + 1\right)$$

(See page 16 of the Notes)

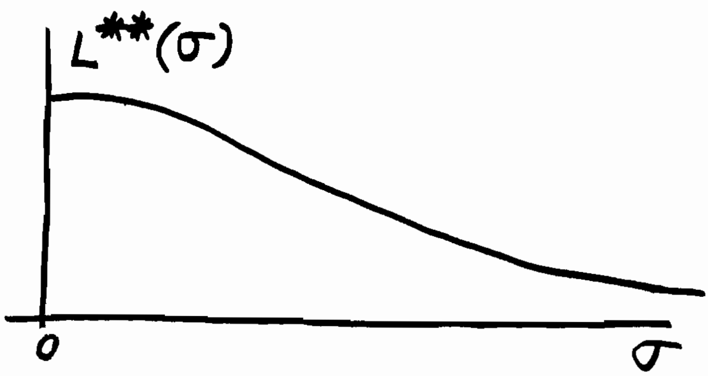
- $t_{\alpha/2:(n-1)}$: upper $\alpha/2$ point of t_{n-1}.

<u>Confidence Sets for σ:</u>

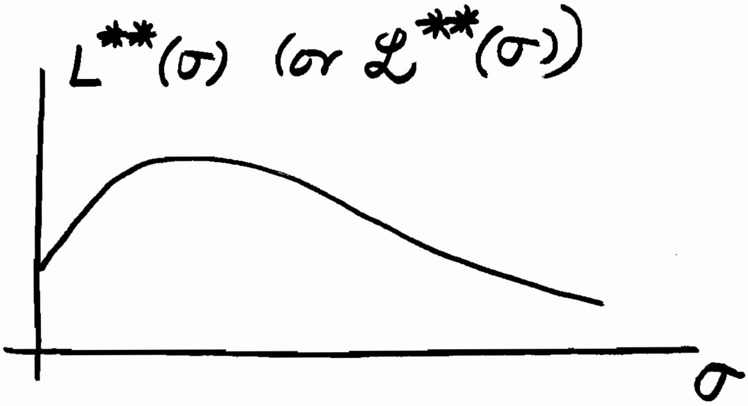
Let $L^{**}(\sigma) = \sup_{\mu} L(\mu,\sigma)$

("profile likelihood for σ")

<u>Range = 0 or 1:</u> Let $\mathcal{L}^{**}(\sigma) = \ln L^{**}(\sigma)$



Range ≥ 2:



As a way of making CI's for σ I'll use

$$\text{Confidence interval for } \sigma = \left\{ \sigma \;\middle|\; \begin{array}{l} L^{**}(\sigma) \text{ is not too much} \\ \text{below maximum possible} \end{array} \right\}$$

Large sample theory says that approximately

$$(1-\alpha) \text{ level CI for } \sigma = \left\{ \sigma \;\middle|\; \mathcal{L}^{**}(\sigma) > \sup_{\sigma} \mathcal{L}^{**}(\sigma) - \tfrac{1}{2}\chi_1^2 \right\}$$

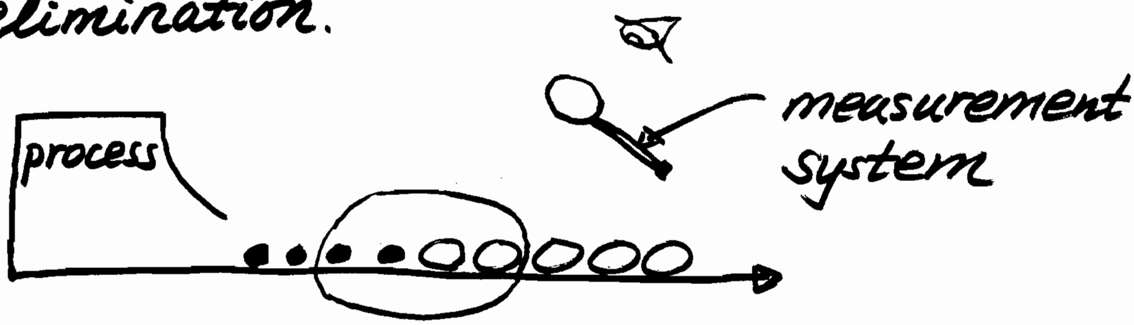
This turns out to work poorly for small n — Johnson Lee has found replacements for χ_1^2 so that the resulting intervals hold nominal confidence level (see §1.7 of the Notes for details).

Start Process Monitoring

Read V&J: $\underbrace{3.1-3.5,\ 4.3,\ 4.4}$

"Shewhart" "Control Charting"

Basic Problem: Process "watching" for purposes of change detection and elimination.



Watch for "lack of physical stability"

Implication: When such is detected some (unspecified) physical intervention is needed (to set things "aright")

"process stability" = ?

Complete lack of variation is too much to hope for.

Classical/Shewhart interpretation:

"process stability" = pattern of variation (over time) is "constant" in the sense that measurements generated "look iid"

Possibly a better/more enlightened/ modern interpretation:

"process stability" = pattern of observed variation is "constant" in the sense that data look like they come from <u>a fixed stochastic model</u> (one that we "expect")

Process monitoring is about <u>change detection</u>.

Original Shewhart "Control Charting" Idea

1) Identify some sensible measure of process performance/behavior to plot against time (Q)
2) Apply some probability theory to identify a sensible "stable process" distribution for Q.
3) Establish upper and lower "control limits" based on 2).

These limits are meant to separate "typical" values of Q from ones that are "rare" if indeed the process is "stable".

— You might get them by

a) Using small lower and upper % points of the dsn in 2)

or (nearly equivalently)

b) $\left.\begin{array}{l} \mu_Q - 3\sigma_Q = LCL_Q \\ \mu_Q + 3\sigma_Q = UCL_Q \end{array}\right\}$ "3 sigma limits"

4) Plot values of Q against time and declare the process "out of control" (and thus in need of physical intervention) at any period where Q falls outside limits from 3).

Various particular Q lead to different famous charts:

$\bar{x}$, R, s, $\tilde{x}$, p, np, c, u, etc.

See 3.2 and 3.3 of V&J — <u>Table 3.11</u> of V&J for a summary of these standard charts. (page 107)

Additional Introductory Qualitative Points:

① Concerning situations where measurements y are processed into a Q

Control limits on Q $\neq$ Engineering Specifications on y

Control limits on Q have to do with assessing process stability (applied to Q)

Engineering Specifications on y have to do with defining functionality/acceptability of individuals

② Standards Given vs. Retrospective Contexts / Analyses

"Standards given" context:

— Before beginning process monitoring one is furnished with sensible values for process parameters and these are used to create control limits that are applied to Q's as they are generated.

— e.g., if "standard values" for μ and σ are $\mu = 5.0$ and $\sigma = 1.715$ and I'm going to do Shewhart $\bar{x}$ charting...

Control limits for $\bar{x}$ (based on $n=5$):

$$UCL_{\bar{x}} = \mu_{\bar{x}} + 3\sigma_{\bar{x}}$$

$$= 5 + 3 \cdot \frac{1.715}{\sqrt{5}} = 7.3$$

$$LCL_{\bar{x}} = \mu_{\bar{x}} - 3\sigma_{\bar{x}}$$

$$= 5 - 3 \cdot \frac{1.715}{\sqrt{5}} = 2.7$$

"Retrospective"/"as past data" context:

Here one is furnished with a data record (from, say, r periods) and is asked:

"Does stability over the period of data collection seem plausible?"

Here the only obvious method of operation is:

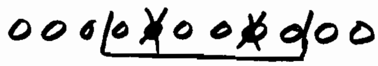
a) temporarily assume stability over the period of study and make provisional estimates of (supposedly constant) parameters.

b) use the estimates from a) to make control limits, apply those retrospectively to the data in hand to criticize the stability assumption.

Jargon for types of process monitoring data:

"Attributes Data" Scenarios (Count Data)

1. % (fraction) nonconforming (defective) context:



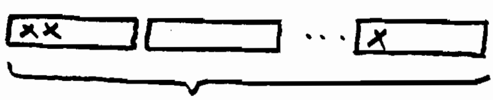
sample of n widgets

items are individually either conforming or nonconforming

X = # nonconforming in sample np charting

$\frac{X}{n} = \hat{p}$ = fraction nonconforming p charting

2. mean nonconformities (defects) per unit context:



k (possibly noninteger) inspection units

Each inspection unit may have multiple nonconformities/defects —

X = total nonconformities

$k=1$ plot X c charting u charting

plot $\frac{X}{k} = \hat{u}$ = nonconformities/unit

"Variables Data" Scenarios (Measurements)

$x_1, x_2, \cdots, x_n$: measurements on a sample of n items

Charts for distributional spread / "process short term variation"

R, s

Charts for process aim/location:

$\bar{x}$ $(\tilde{x})$

Usual control limits for these come from normal distribution theory.

<u>Example</u> $n=5$ $\bar{x}, R$

<u>Standards Given</u> $(\mu=5,\ \sigma=1.715)$

<u>$\bar{x}$ chart limits</u>:

$$UCL_{\bar{x}} = \mu_{\bar{x}} + 3\sigma_{\bar{x}} = 5 + 3 \cdot \frac{1.715}{\sqrt{5}}$$

$$LCL_{\bar{x}} = \mu_{\bar{x}} - 3 \cdot \sigma_{\bar{x}} = 5 - 3 \cdot \frac{1.715}{\sqrt{5}}$$

<u>R chart limits</u>:

$\mu_R \leftarrow$ "center line" for R chart center line $= d_2(5) \cdot \sigma = 2.326(1.715)$

$= 3.99$

$UCL_R = \mu_R + 3\sigma_R = d_2(5)\cdot\sigma + 3\cdot d_3(5)\cdot\sigma$

$= \sigma\,\underbrace{(d_2(5) + 3d_3(5))}_{D_2(5)}$

$= 1.715\,(4.918)$

$= 8.43.$

LCL_R

$= \mu_R - 3\sigma_R$

$= D_1(n)\cdot\sigma$

$= 0.$

$d_2(n)$, $d_3(n)$, $D_2(n)$, etc.: see Table A.1 on page 509 of V&J (the text).