Read for Context Ch 1.9 of V&J (Text)
Six Sigma Handout

Begin by reading Appendix, Ch 1 of V (Notes)
Ch 2 of V&J
Vardeman IIE Trans. '99
Vardeman & VanValkenburg Technometrics '99

Introduction/What's the course about?

$\boxed{\text{Statistical}}$ Methods and Theory useful in production of $\boxed{\text{Quality}}$ Goods and Services

statistics is the study of how best to

1) collect
2) summarize

and 3) draw conclusions from data

all recognizing the reality of variation

Relevance to QA? (Quality Assurance)

quality $\longleftrightarrow$ fitness for use

quality of design

features
species

quality of conformance

small <u>variation</u> from ideal design

To assure quality of conformance, you need to observe what's going on (and intervene appropriately)

— i.e., you must use data/statistics

variation $\approx$ unquality

A useful approximation from probability theory (see Appendix of the Notes)

(Multivariate Δ method)

(This is the origin of (1.3) of the Notes and Section 5.4 of the Text.)

$$\underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} \qquad \underset{\sim}{\mu} = \begin{pmatrix} EX_1 \\ EX_2 \\ \vdots \\ EX_p \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix}$$

$$\underset{p \times p}{\underset{\sim}{\Sigma}} = \begin{bmatrix} \operatorname{Var} X_1 & \operatorname{cov}(X_1, X_2) & \operatorname{Cov}(X_1, X_3) & \cdots \\ \operatorname{cov}(X_1, X_2) & \operatorname{Var} X_2 & & \\ & & \ddots & \\ & & & \operatorname{Var} X_p \end{bmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & & \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \sigma_p^2 \end{pmatrix} = (\rho_{ij}\sigma_i\sigma_j)$$

<u>Fact:</u> If $\underset{k \times p}{\underset{\sim}{A}} = (a_{ij})$ then for the random vector $\underset{k\times 1}{\underset{\sim}{Y}} = \underset{k\times p}{\underset{\sim}{A}}\,\underset{p\times 1}{\underset{\sim}{X}}$

mean vector for $\underset{\sim}{Y}$ $= \begin{pmatrix} EY_1 \\ \vdots \\ EY_k \end{pmatrix} = \underset{\sim}{A}\,\underset{\sim}{\mu}$

Variance-covariance matrix for $\underset{\sim}{Y}$ $= \underset{k\times p}{\underset{\sim}{A}}\,\underset{p\times p}{\underset{\sim}{\Sigma}}\,\underset{p\times k}{\underset{\sim}{A}'}$

<u>Note</u>: The $k=1$ version of this with uncorrelated X_i is the standard

$$\operatorname{Var} Y = \sum a_{ii}^2 \sigma_i^2$$

<u>The (multivariate) propagation of error / delta method</u> – This says

$$Y_1 = g_1(\underset{\sim}{X})$$
$$Y_2 = g_2(\underset{\sim}{X})$$
$$\vdots$$
$$Y_k = g_k(\underset{\sim}{X})$$

$$\underset{k\times 1}{\underset{\sim}{Y}} = \begin{pmatrix} g_1(\underset{\sim}{X}) \\ g_2(\underset{\sim}{X}) \\ \vdots \\ g_k(\underset{\sim}{X}) \end{pmatrix}$$

How about approximations for mean and variance-covariance of $\underset{\sim}{Y}$?

If the g_i are smooth (i.e., have derivatives), let

$$\underset{k\times p}{\underset{\sim}{D}} = \left(\left. \frac{\partial g_i}{\partial x_j} \right|_{\mu_1, \mu_2, \ldots, \mu_p} \right).$$

Then a multivariate Taylor approximation says that if each x_i is near μ_i

$$\underset{\sim}{y} = \begin{pmatrix} g_1(\underset{\sim}{x}) \\ g_2(\underset{\sim}{x}) \\ \vdots \\ g_k(\underset{\sim}{x}) \end{pmatrix} \approx \begin{pmatrix} g_1(\underset{\sim}{\mu}) \\ g_2(\underset{\sim}{\mu}) \\ \vdots \\ g_k(\underset{\sim}{\mu}) \end{pmatrix} + \underset{\sim}{D}\,(\underset{\sim}{x} - \underset{\sim}{\mu})$$

$$\underset{\sim}{Y} = \begin{bmatrix} g_1(\underset{\sim}{X}) \\ \vdots \\ g_k(\underset{\sim}{X}) \end{bmatrix} \approx \begin{bmatrix} g_1(\underset{\sim}{\mu}) \\ \vdots \\ g_k(\underset{\sim}{\mu}) \end{bmatrix} + \underset{\sim}{D}\,(\underset{\sim}{X} - \underset{\sim}{\mu})$$

So I might hope that

mean vector for $\underset{\sim}{Y}$ $\approx \begin{pmatrix} g_1(\underset{\sim}{\mu}) \\ \vdots \\ g_k(\underset{\sim}{\mu}) \end{pmatrix}$

Variance-covariance matrix for $\underset{\sim}{Y}$ $\approx \underset{\sim}{D}\underset{\sim}{\Sigma}\underset{\sim}{D}'$

Statistics and Measurement

Basic concepts: validity, precision, accuracy — pages 17–19 (V+J)

Page 19 V+J (Text) Basic Measurement Model

$$y = \overset{\text{fixed}}{x} + \varepsilon$$

y: measured or observed value

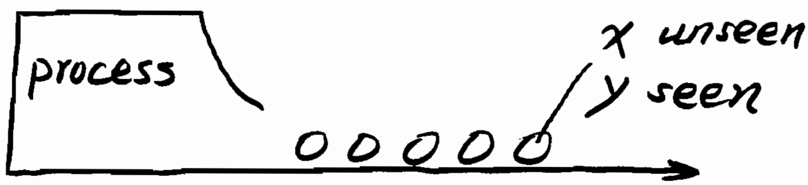
x: Quantity of interest

ε: measurement error, mean β, variance $\sigma^2_{\text{measurement}}$

$$Ey = E(x+\varepsilon) = E(x) + E(\varepsilon)$$
$$= x + \beta \leftarrow \text{bias}$$

$$\mathrm{Var}\, y = \mathrm{Var}(x+\varepsilon) = \mathrm{Var}\,\varepsilon = \sigma^2_{\text{measurement}}$$

But now suppose that x is random—perhaps this models item-to-item variation



$$Ex = \mu_x \qquad \mathrm{Var}\, x = \sigma_x^2$$

What are properties of y?

$$Ey = E(x+\varepsilon) = Ex + E\varepsilon$$
$$= \mu_x + \beta$$

$$\mathrm{Var}\, y = \mathrm{Var}(x+\varepsilon) = \mathrm{Var}\, x + \mathrm{Var}\,\varepsilon$$
$$= \sigma_x^2 + \sigma^2_{\text{measurement}}$$

x, ε uncorrelated

How to "get at" σ_x^2?

Elementary Suggestion:

approximates Var Y

$$\left.\begin{array}{lcl} \text{widget \#1} & \longrightarrow & y_1 \\ & & y_2 \\ \vdots & & \vdots \\ \text{widget \#}n & \longrightarrow & y_n \end{array}\right\} \longrightarrow S_y^2$$

approximates $\sigma^2_{\text{measurement}}$

$$\text{widget \#}n{+}1 \begin{array}{l} \nearrow \\ \\ \searrow \end{array} \left.\begin{array}{l} y_1' \\ \vdots \\ y_m' \end{array}\right\} \longrightarrow s^2$$

Try $\quad \hat{\sigma}_x = \sqrt{\max\left(0,\, S_y^2 - s^2\right)}$

How good is this? What are alternatives?

What about confidence intervals (C.I.'s) for σ_x, etc.?

Need to "review" some probability (see first few sections of the Notes) (Vardeman)

Let $X_1, \cdots, X_n$ iid $N(\mu, \sigma^2)$

$$s^2 = \frac{1}{n-1} \sum (X_i - \overline{X})^2$$

$$R = \max X_i - \min X_i$$

$$Z_i = \frac{X_i - \mu}{\sigma} \leftarrow \text{iid } N(0,1)$$

$$s^2 = \sigma^2 \cdot (\text{sample variance of } Z_i\text{'s})$$

$$R = \sigma \cdot (\text{sample range of } Z_i\text{'s})$$

So probability facts with the standard normal distribution allow us to develop the following about s^2, R:

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{and thus}$$

$$Es^2 = \sigma^2 \qquad \operatorname{Var} s^2 = \frac{2\sigma^4}{n-1}$$

$$Es = E\sqrt{s^2} = \sigma \cdot c_4(n)$$

$$\sqrt{\operatorname{Var} s} = \sigma \cdot c_5(n)$$

$$\sqrt{\operatorname{Var} s} = \sqrt{Es^2 - (Es)^2}$$

a "control chart constant"

$$c_5(n) = \sqrt{1 - c_4^2(n)}$$

$W = \frac{R}{\sigma} \sim$ something computable (see top of page 3 of the Notes)

$$ER = \sigma \cdot EW = \sigma \cdot \underbrace{d_2(n)}$$

$$\sqrt{\mathrm{Var}\, R} = \sigma \cdot \underbrace{d_3(n)}$$

famous tabled control chart constants

These facts give us means of thinking about $\hat{\sigma}_x$ and many other related estimates — e.g., let

$$g(t_1, t_2) = \sqrt{\max(0, t_1 - t_2)}$$

note that $\hat{\sigma}_x = g(s_y^2, s^2)$

and we might try the propagation of error / delta method formulas as means of evaluating the effectiveness of $\hat{\sigma}_x$.

Model S_y^2 and S^2 as independent sample variances (from normal universes) and note that

$$E(S_y^2 - S^2) = ES_y^2 - ES^2 = \sigma_x^2 > 0$$

So near $t_1 = ES_y^2$, $t_2 = ES^2$ the maximum in formula for g doesn't come into play. Thus I can compute partial derivatives and use propagation of error on $\sqrt{t_1 - t_2}$.

$$E\hat{\sigma}_x \approx \sigma_x$$

$$\mathrm{Var}\,\hat{\sigma}_x \approx \left(\frac{\partial g}{\partial t_1}\Big|_{ES_y^2,\, ES^2}\right)^2 \mathrm{Var}\,S_y^2$$

$$+ \left(\frac{\partial g}{\partial t_2}\Big|_{ES_y^2,\, ES^2}\right)^2 \mathrm{Var}\,S^2$$

$$= \left(\frac{\frac{1}{2}(1)}{\sqrt{\sigma_x^2}} \right)^2 \cdot \frac{2(\sigma_x^2 + \sigma^2_{measurement})^2}{(n-1)}$$

$$+ \left(\frac{\frac{1}{2}(-1)}{\sqrt{\sigma_x^2}} \right)^2 \cdot \frac{2\,\sigma^4_{measurement}}{(m-1)}$$

Useful of this?

Note first that this is useful pre-data for planning purposes.

approx. $\mathrm{Var}\,\hat{\sigma}_x = h(\sigma_x^2, \sigma^2_{meas.}, n, m)$

So if I have "worst case" values for $\sigma_x^2, \sigma^2_{measurement}$, I may fiddle around with n, m and see what is needed in order to make this acceptably small.

Note also that this approximation can be used post-data (i.e., with n, m chosen, s_y^2 and s^2 in hand) to find a standard error for $\hat{\sigma}_x$.

$$\underbrace{\sqrt{h(s_y^2 - s^2, s^2, n, m)}} = \sqrt{\widehat{\text{approx. Var}\,\hat{\sigma}_x}}$$

a post-data indication of how well I know σ_x.

This can be taken a step further and put into a setting where I want to <u>estimate</u> not just σ_x but

<u>σ_x and $\sigma_{\text{measurement}}$.</u>

i.e., with $f(t_1, t_2) = \sqrt{t_2}$

I note that $s = f(s_y^2, s^2)$,

and I observe that an interesting random vector is

$$\underbrace{\begin{pmatrix} \hat{\sigma}_x \\ s \end{pmatrix}}_{\underset{\sim}{Y}} = \begin{pmatrix} g(s_y^2, s^2) \\ f(s_y^2, s^2) \end{pmatrix}$$

and I can apply the (multivariate) propagation of error method to get an approximate mean and variance-covariance matrix for $\underset{\sim}{Y}$.

$$E\begin{pmatrix} \hat{\sigma}_x \\ s \end{pmatrix} \approx \begin{pmatrix} \sigma_x \\ \sigma_{\text{measurement}} \end{pmatrix}$$

Note that if

$$\frac{\nu s^2}{Es^2} \sim \chi^2_\nu \longleftarrow \text{variance } 2\nu$$

then

$$\text{Var}\, s^2 = \text{Var}\left[\left(\frac{Es^2}{\nu}\right)\left(\frac{\nu s^2}{Es^2}\right)\right]$$

$$= \left(\frac{Es^2}{\nu}\right)^2 \cdot 2\nu = \frac{2(Es^2)^2}{\nu} = \frac{2\sigma^4_{\text{meas.}}}{\nu}$$