Stat/IE 531 Midterm Exam Solutions March 27, 2007

1. (a) $\hat{x} = \frac{y - 0.00615}{0.00722} = \frac{1.200 - 0.00615}{0.00722} \approx 165.353.$

(b) Using the delta method, we have an approximate 90% confidence interval

$$\hat{x}_{n+1} \pm t \cdot \frac{\sqrt{MSE}}{|b_1|} \cdot \sqrt{1 + \frac{1}{n} + \frac{(\hat{x}_{n+1} - \bar{x})^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

i.e., $165.353 \pm 1.684 \cdot \frac{0.002171}{0.00722}$ $(t_{0.95,38} \approx 1.684)$

$$\times \sqrt{1 + \frac{1}{40} + \frac{(165.353 - 157.5)^2}{299250}}$$

or 165.353 ± 0.513

2. (a) $UCL_X = \lambda + 3\sqrt{\lambda} = 1 + 3 \cdot \sqrt{1} = 4$

$LCL_X = \lambda - 3\sqrt{\lambda} = 1 - 3 \cdot \sqrt{1} = -2 \Rightarrow LCL_X = 0.$

(b) $q = P[X_1$ plots outside control limits$]$

$$= P(X_1 > 4] = 1 - P[X_1 \le 4] = 1 - \sum_{k=0}^{4} \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= 1 - \left(\sum_{k=0}^{4} \frac{1}{k!}\right) \cdot e^{-1} \approx 0.00366.$$

$ARL = \frac{1}{q} \approx 273.2.$

(c) $ARL = \frac{1}{q} \ge 500 \Rightarrow q \le 0.002$. That is,

$P[X_1 > UCL_X] \le 0.002 \Rightarrow UCL_X \ge 5$ because

$P[X_1 > 4] > 0.002$ and $P[X_1 > 5] \approx 0.00059 < 0.002$.

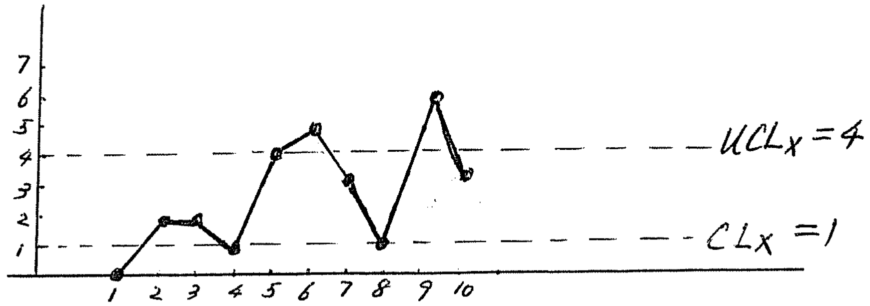
Thus, we should use $LCL_X = 0$ and $UCL_X = 5$.

(d) When $\lambda = 2$, $q = P[X_1$ plots outside control limits$]$

$$= P[X_1 > 4] = 1 - \left(\sum_{k=0}^{4} \frac{2^k}{k!}\right) \cdot e^{-2} \approx 0.05265$$

Thus, $ARL = \frac{1}{q} \approx 18.99$.

(e)



Yes. There are two alarms signaled at units 6 and 9.

3. (a) Here, $u = v = 0$, $\delta = 0.3$. Thus, $k_1^{opt} = \mu_{\bar{X}} + \frac{\delta}{2} = \mu + \frac{\delta}{2}$

$$= 100 + \frac{0.3}{2} = 100.15, \text{ and } k_2^{opt} = \mu_{\bar{X}} - \frac{\delta}{2} = 100 - \frac{0.3}{2} = 99.85.$$

Also, $h = \mathcal{H} \cdot \sigma_{\bar{X}} = \mathcal{H} \cdot \frac{\sigma}{\sqrt{n}}$, where $\mathcal{H}$ is obtained from Table 4.6 of V&J (p. 147). Noting that $\mathcal{K} = \frac{k_1^{opt} - \mu_{\bar{X}}}{\sigma_{\bar{X}}}$

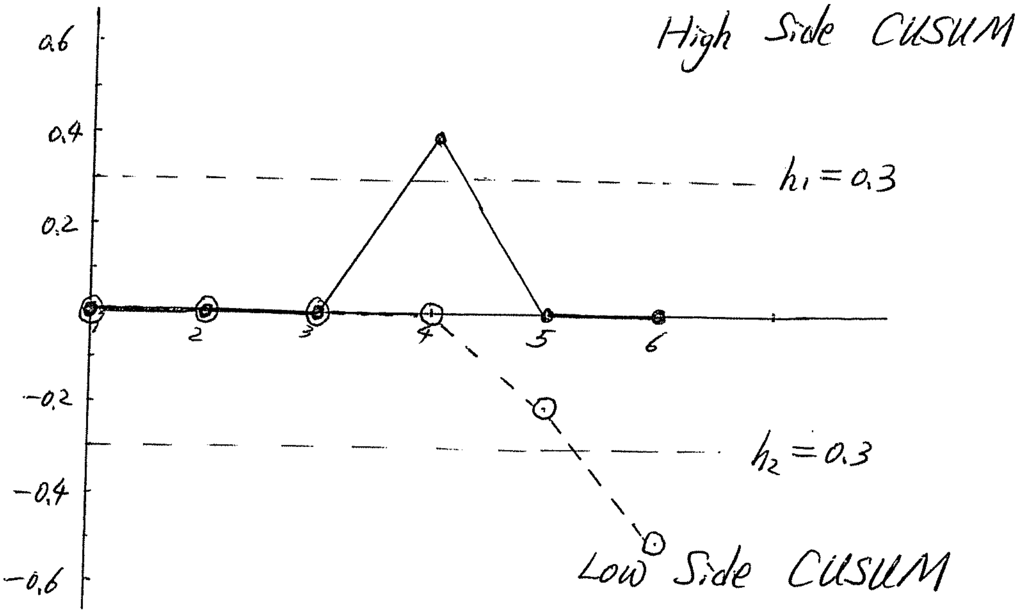
$$= \frac{(\mu + \frac{\delta}{2}) - \mu}{\sigma / \sqrt{n}} = \frac{\frac{\delta}{2} \cdot \sqrt{n}}{\sigma} = \frac{\frac{0.3}{2} \cdot \sqrt{5}}{0.2} \approx 1.68,$$ and using

$\mathcal{K} = 1.50$ in Table 4.6, we have $\mathcal{H} = 1.71$. (An extrapolation is also OK.) Thus, $h = \mathcal{H} \cdot \frac{\sigma}{\sqrt{n}} = (1.71) \cdot \frac{0.2}{\sqrt{5}} \approx 0.15$.

(b) The high side and low side CUSUM sequences are given below.

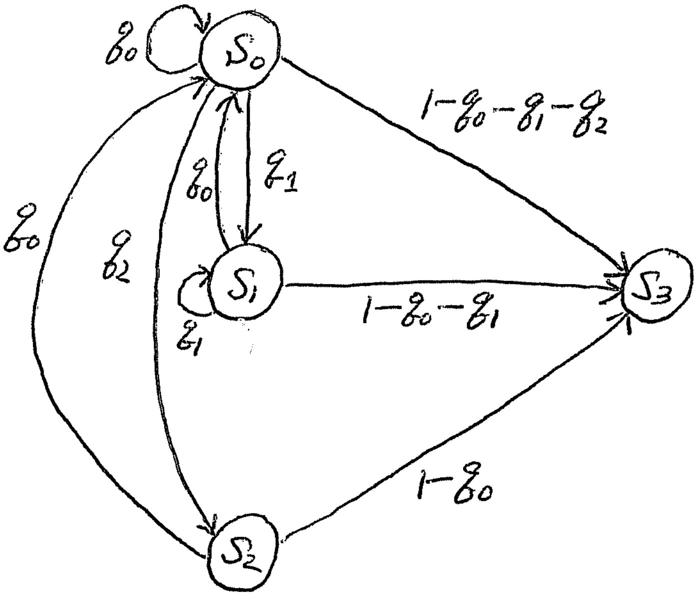
i	$\bar{X}_i$	U_i	V_i
1	100.2	$\max(0, 100.2-100.2+0) = 0$	$\min(0, 100.2-99.8+0) = 0$
2	99.9	$\max(0, 99.9-100.2+0) = 0$	$\min(0, 99.9-99.8+0) = 0$
3	99.8	$\max(0, 99.8-100.2+0) = 0$	$\min(0, 99.8-99.8+0) = 0$
4	100.6	$\max(0, 100.6-100.2+0) = 0.4$	$\min(0, 100.6-99.8+0) = 0$
5	99.6	$\max(0, 99.6-100.2+0.4) = 0$	$\min(0, 99.6-99.8+0) = -0.2$
6	99.5	$\max(0, 99.5-100.2+0) = 0$	$\min(0, 99.5-99.8-0.2) = -0.5$

(Note that $U_i = \max(0, \bar{X}_i - k_1 + U_{i-1})$ and $V_i = \min(0, \bar{X}_i - k_2 + V_{i-1})$)



Yes. Alarms are signaled at subgroups 4 (high side CUSUM) and 6 (low side CUSUM).

4. Let S_i = "no alarm yet and current Q is i" for $i=0,1,2$ and S_3 = "alarm". Then we have the following Markov Chain diagram:



The transition matrix is

$$P_{4\times4} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc} S_0 & S_1 & S_2 & S_3 \\ q_0 & q_1 & q_2 & 1-\sum_{i=0}^{2} q_i \\ q_0 & q_1 & 0 & 1-q_0-q_1 \\ q_0 & 0 & 0 & 1-q_0 \\ 0 & 0 & 0 & 1 \end{array}\right]\begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ S_3 \end{array}$$

The desired ARL is the mean number of steps from S_0 to S_3, i.e., L_1, where L_1 is the first element of $L=(I-R)^{-1}\underline{1}$, where $R=\begin{pmatrix} q_0 & q_1 & q_2 \\ q_0 & q_1 & 0 \\ q_0 & 0 & 0 \end{pmatrix}$.