Stat/IE 531 Final Exam Solutions Spring 2007

1. (a) Here $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$, $\delta = 0.3$ and thus shift $= \frac{\delta}{\sigma_{\bar{X}}}$

$= \frac{0.3}{0.2/\sqrt{5}} \approx 3.35$. Then by Table 4.2, $\lambda^{opt} \approx 0.78$,

and by Table 4.3, $K = 3$. We thus have

$$UCL_{EWMA} = \mu_{\bar{X}} + K \cdot \sigma_{\bar{X}} \cdot \sqrt{\frac{\lambda^{opt}}{2-\lambda^{opt}}} = 100 + 3 \cdot \frac{0.2}{\sqrt{5}} \sqrt{\frac{0.78}{2-0.78}}$$

$$\approx 100.215$$

$$LCL_{EWMA} = \mu_{\bar{X}} - K \cdot \sigma_{\bar{X}} \cdot \sqrt{\frac{\lambda^{opt}}{2-\lambda^{opt}}} \approx 99.785.$$

(b) Now we have $D^* = \dfrac{\left|\mu_{\bar{X}} - \frac{UCL_{EWMA} + LCL_{EWMA}}{2}\right|}{\sigma_{\bar{X}}}$

$$= \frac{|101 - 100|}{0.3/\sqrt{5}} \approx 7.45$$

and $K^* = \dfrac{UCL_{EWMA} - LCL_{EWMA}}{2\sigma_{\bar{X}}} \cdot \sqrt{\dfrac{2-\lambda^{opt}}{\lambda^{opt}}} = \dfrac{K \cdot \sigma^0_{\bar{X}}}{\sigma_{\bar{X}}} = \dfrac{3 \times \frac{0.2}{\sqrt{5}}}{\frac{0.3}{\sqrt{5}}} = 2.$

From Table A.3, we have the ARL to be less than 1.2.

(c) With $EWMA_0 = 100$ and $\lambda = 0.3$, we have

$$EWMA_1 = 0.3 \cdot (100.2) + 0.7 \cdot (100) = 100.06$$

$$EWMA_2 = 0.3 \cdot (99.9) + 0.7 (100.06) = 100.012$$

$EWMA_3 \approx 99.9484$, $EWMA_4 \approx 100.144$, $EWMA_5 \approx 99.981$, and $EWMA_6 \approx 99.837$. Thus, none of the subgroups produced out-of-control alarms.

2. (a) We have $Z(t) = Y(t) - \sum_{s=0}^{t-1} \Delta X(s) = Y(t) - \sum_{s=0}^{t-1} E(s)$

$$= Y(t) + \sum_{s=0}^{t-1} Y(s) = \sum_{s=0}^{t} Y(s) = \sum_{s=0}^{t} e(s),$$

where $e(s) = Y(s)$ $(s \geq 0)$ are iid $N(0, \sigma^2)$ random variables.

(b) Now, $Y(t) = Z(t) + \sum_{s=0}^{t-1} \Delta X(s) = \sum_{s=0}^{t} e(s) - 0.5 \sum_{s=0}^{t-1} Y(s)$

for $t \geq 0$. That is,

$$Y(t-1) = \sum_{s=0}^{t-1} e(s) - 0.5 \sum_{s=0}^{t-2} Y(s) \quad \text{for } t \geq 1.$$

Then, $Y(t) - Y(t-1) = e(t) - 0.5\, Y(t-1)$ for $t \geq 1$

or $Y(t) - 0.5\, Y(t-1) = e(t)$ for $t \geq 1$.

3. (a) $$\hat{C}_{pk} = \frac{U - L - 2\left|\bar{x} - \frac{U+L}{2}\right|}{6s} = \frac{10.005 - 9.995 - 2 \cdot |9.9996 - 10|}{6 \times 0.00369}$$

≈ 0.4155.

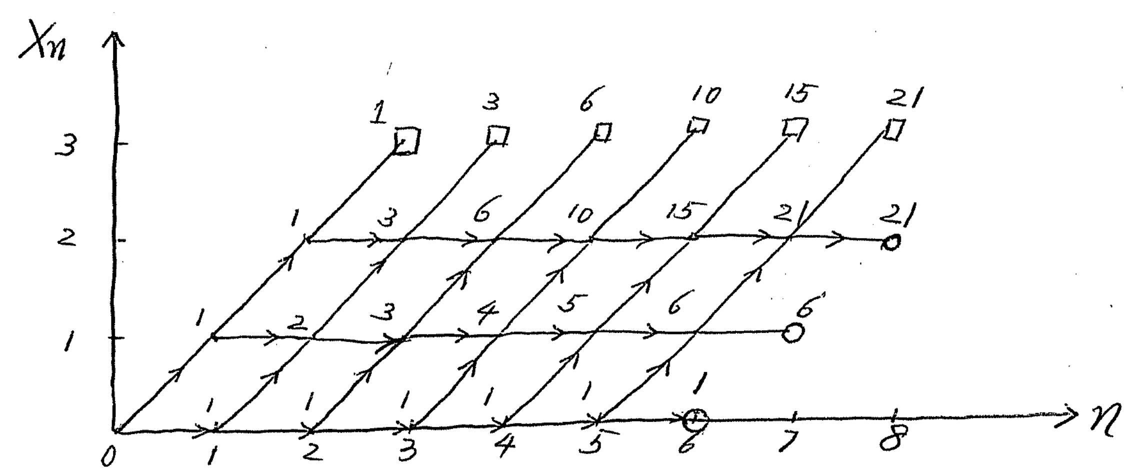
A 95% lower confidence bound for C_{pk} is

$$\hat{C}_{pk} - z \cdot \sqrt{\frac{1}{9n} + \frac{\hat{C}_{pk}^2}{2n-2}} \approx 0.4155 - 1.645 \cdot \sqrt{\frac{1}{9\times5} + \frac{0.4155^2}{2\times5-2}}$$
$$\approx 0.0712.$$

(see (5.10) of V&J).

(b) From Table A.9b, we have a 95% upper tolerance bound for 99% of all diameter measurements given by $\bar{x} + \tau_1 \cdot s = 9.9996 + 5.741 \times 0.00369 \approx 10.0208$.

4. (a) $\square$: reject o: accept



$$OC = P_a(p) = \sum_{(n, X_n) \in A} \begin{pmatrix} \text{path count} \\ \text{from } (0,0) \text{ to } (n, X_n) \end{pmatrix} \cdot p^{X_n} (1-p)^{n-X_n}$$

$$= 1 \cdot p^0 \cdot (1-p)^6 + 6 \cdot p^1 \cdot (1-p)^6 + 21 \cdot p^2 \cdot (1-p)^6$$

$$= (1 + 6p + 21p^2) \cdot (1-p)^6.$$

$$ASN = \sum_{(n, X_n) \in A \cup R} n \cdot P_r(\text{ending at } (n, X_n))$$

$$= 3 \cdot p^3 \cdot (1-p)^0 + 4 \cdot 3 \cdot p^3 (1-p)^1 + 5 \cdot 6 \cdot p^3 (1-p)^2$$
$$+ 6 \cdot \left(1 \cdot p^0 (1-p)^6 + 10 \cdot p^3 (1-p)^3\right) + 7 \cdot \left(6 \cdot p^1 (1-p)^6\right.$$
$$\left. + 15 \cdot p^3 \cdot (1-p)^4\right) + 8 \cdot \left(21 \cdot p^2 (1-p)^6 + 21 \cdot p^3 (1-p)^5\right)$$

$$= p^3 \cdot \left(3 + 12(1-p) + 30(1-p)^2 + 60(1-p)^3 + 105(1-p)^4 + 168(1-p)^5\right)$$
$$+ (1-p)^6 \cdot (6 + 42p + 168p^2).$$

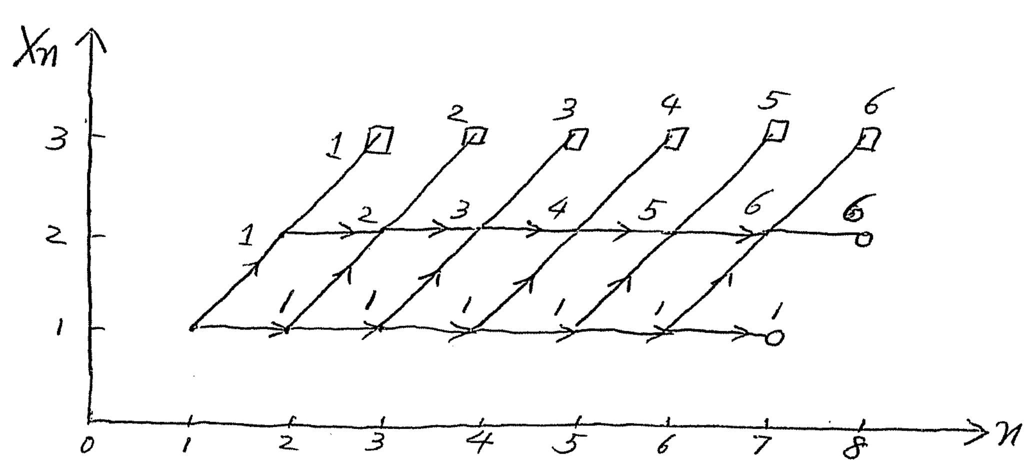
(b) $$AOQ = \sum_{(n, X_n) \in A} \left(1 - \frac{n}{N}\right) \cdot p \cdot P_r\left(\text{ending at } (n, X_n)\right)$$

$$= \left(1 - \frac{6}{100}\right) \cdot p \cdot (1-p)^6 + \left(1 - \frac{7}{100}\right) \cdot 6p^2 (1-p)^6 + \left(1 - \frac{8}{100}\right) \cdot 21 p^3 (1-p)^6.$$

$$ATI = N \cdot (1 - P_a) + \sum_{(n, X_n) \in A} n \cdot P_r\left(\text{ending at } (n, X_n)\right)$$

$$= 100 \cdot (1 - P_a(p)) + 6 \cdot (1-p)^6 + 42 \cdot p(1-p)^6 + 168 \cdot p^2 (1-p)^6.$$

(c)



Stop Sampling Point	UMVUE, $\hat{p}$
(6, 0)	0
(7, 1)	1/6
(8, 2)	6/21 = 2/7
(3, 3)	1/1 = 1
(4, 3)	2/3
(5, 3)	3/6 = 1/2
(6, 3)	4/10 = 2/5
(7, 3)	5/15 = 1/3
(8, 3)	6/21 = 2/7

$$\hat{p} = \frac{\text{path count from } (1,1) \text{ to } (n, X_n)}{\text{path count from } (0,0) \text{ to } (n, X_n)}$$

5. Consider a perspective B scenario only here.

The conditional distributions for $p|X=x$ are given by $f(p|x) = \dfrac{f(x|p)\cdot f(p)}{\sum_p f(x|p)\cdot f(p)}$

$$= \frac{\binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{1}{3}}{\sum_p \binom{n}{x} p^x (1-p)^{n-x} \cdot \frac{1}{3}} = \frac{p^x (1-p)^{n-x}}{\sum_p p^x (1-p)^{n-x}}$$ for $p = 0, 0.1, 0.2$ and $x = 0, \cdots, n$.

Thus, $E(p|x) = \sum_p f(p|x) = \dfrac{\sum_p p^{1+x}(1-p)^{n-x}}{\sum_p p^x(1-p)^{n-x}}$.

That is, $E(p|0) = \dfrac{\sum_p p\cdot(1-p)^n}{\sum_p (1-p)^n} = \begin{cases} 0.1 & n=0 \\ \frac{5}{54} & n=1 \\ \frac{209}{2430} & n=2 \end{cases}$

Note that $E(p|0) > \frac{k_1}{k_2}$ $(=\frac{1}{12})$ for $n=0,1,2$.

Thus, for each sample size $n=0,1,2$, the best fixed n inspection plan would reject the lot.

The sample size $(n=0,1,2)$ happens to be equally good as long as the inspection plan is to reject the lot regardless of the outcomes of inspection (i.e., inspect all).