Stat/IE 531 Midterm Exam Solutions March 24, 2005

1. Here, $I=10$, $J=2$, and $m=3$. Also, $MSA=10$, $MSB=12$, $MSAB=6$, $MSE=2$, $df_A=I-1=9$, $df_B=1$, $df_{AB}=(I-1)(J-1)=9$, and $df_{Error}=40$.

(a) $\hat{\sigma}_{repeatability}=\sqrt{MSE}=\sqrt{2}=1.414$.

$\hat{\sigma}_{reproducibility}=\sqrt{\hat{\sigma}_\beta^2+\hat{\sigma}_{\alpha\beta}^2}=\sqrt{\max(0,\ \frac{1}{mI}MSB+(1-\frac{1}{I})\frac{1}{m}MSAB-\frac{1}{m}MSE)}$

$=\sqrt{\max(0,\ \frac{12}{30}+\frac{9}{30}\times 6-\frac{2}{3})}=\sqrt{\frac{23}{15}}\doteq\sqrt{1.533}\doteq 1.238$.

(b) For s.e. of $\hat{\sigma}_{repeatability}$, consider $g(t)=\sqrt{t}$ and $\frac{\partial g(t)}{\partial t}=\frac{1}{2\sqrt{t}}$. Thus $\sqrt{\widehat{Var}(\hat{\sigma}_{repeatability})}=\sqrt{\left(\frac{1}{2\sqrt{MSE}}\right)^2\cdot\frac{2(MSE)^2}{(m-1)\cdot IJ}}$

$=\sqrt{\frac{1}{4\times 2}\cdot\frac{2\cdot(2)^2}{40}}=\sqrt{\frac{1}{40}}\doteq 0.158$.

For s.e. of $\hat{\sigma}_{reproducibility}$, consider $g(t_1,t_2,t_3)$

$=\sqrt{\frac{1}{mI}t_1+\frac{1}{m}(1-\frac{1}{I})t_2-\frac{1}{m}t_3}$. Then $g(MSB, MSAB, MSE)$

$=\hat{\sigma}_{reproducibility}$. Note that $\frac{\partial g}{\partial t_1}=\frac{\frac{1}{mI}}{2g}=\frac{1}{2mI}\cdot\frac{1}{g}$,

$\frac{\partial g}{\partial t_2}=\frac{1}{2m}(1-\frac{1}{I})\cdot\frac{1}{g}$ and $\frac{\partial g}{\partial t_3}=-\frac{1}{2m}\cdot\frac{1}{g}$.

Thus, $\sqrt{\widehat{Var}(\hat{\sigma}_{reproducibility})}=\sqrt{\frac{1}{4\hat{\sigma}^2_{reproducibility}}}\times$

$\sqrt{\left(\frac{1}{mI}\right)^2\cdot\frac{2(MSB)^2}{J-1}+\left[\frac{1}{m}(1-\frac{1}{I})\right]^2\cdot\frac{2(MSAB)^2}{(I-1)(J-1)}+\frac{1}{m^2}\cdot\frac{2(MSE)^2}{(m-1)\cdot IJ}}$

$$=\sqrt{\frac{1}{2\times(23/15)}\cdot\frac{1}{30^2}\left[\frac{12^2}{1}+\frac{9^2\cdot6^2}{9}+\frac{10^2\cdot2^2}{40}\right]}=\sqrt{\frac{239}{1380}}\doteq 0.416.$$

2. (a) The retrospective control limits for the subgroup means are:

$$UCL_{\bar{x}}=\bar{\bar{x}}+A_3(5)\bar{s}=\frac{173.0}{15}+(1.427)\cdot\frac{26.18}{15}=14.024$$

$$LCL_{\bar{x}}=\bar{\bar{x}}-A_3(5)\bar{s}=\frac{173.0}{15}-(1.427)\cdot\frac{26.18}{15}=9.043.$$

(b) The retrospective control limits for the subgroup standard deviations are:

$$UCL_s=B_4(5)\bar{s}=(2.089)\cdot\frac{26.18}{15}=3.646$$

$$LCL_s=B_3(5)\bar{s}=0.$$

(c) The cutting process seems to be stable become all the $\bar{x}$ and $\bar{s}$ values fall within their respective control limits.

(d) (i) $p=P(9.043<\bar{X}<14.024)=P\left(\frac{9.043-10}{1.5/\sqrt{5}}<Z<\frac{14.024-10}{1.5/\sqrt{5}}\right)$

$$=P(-1.43<Z<6.00)=1-0.0764$$

$$ARL=\frac{1}{1-p}=\frac{1}{0.0764}\doteq 13.09.$$

(ii) $p=P(0<S<3.646)=P\left(0<\frac{(n-1)S^2}{\sigma^2}<\frac{(n-1)\cdot(3.646)^2}{\sigma^2}\right)$

$$=P(a\ \chi^2_4\ \text{r.v.}<23.633)=F_{\chi^2_4}(23.633)$$

$$ARL = \frac{1}{1-F_{\chi_4^2}(23.633)}$$

(iii) Because $\bar{x}$ and s are independent, we have

$$p = P(9.043 < \bar{X} < 14.024) \cdot P(0 < S < 3.646)$$
$$= (1-0.0764) \cdot F_{\chi_4^2}(23.633)$$

Thus, $ARL = \frac{1}{1-p} = \left(1-0.9236 \cdot F_{\chi_4^2}(23.633)\right)^{-1}$.

3. (a) Here, all-OK $ARL = 370$ and off-target $ARL = 2.5$.

and $\delta = 8$. From Table 4.8, we have

$$\frac{\delta}{\sigma_{\bar{x}}} = \sqrt{n}\,\frac{\delta}{\sigma} = \sqrt{n} \cdot \frac{8}{10} = 2.37 \Rightarrow n = \left(\frac{10}{8} \times 2.37\right)^2 \doteq 8.78.$$

Thus, $n = 9$. $u = v = 0$. $k_1^{opt} = \mu + \frac{\delta}{2} = 100 + \frac{8}{2} = 104$.

$k_2^{opt} = \mu - \frac{\delta}{2} = 96$. $K = \frac{k_1^{opt} - \mu}{\sigma/\sqrt{n}} = \frac{104-100}{10/\sqrt{9}} = 1.2$.

From Table 4.6, we have $H = 0.2 \times 2.52 + 0.8 \times 1.99 = 2.096$.

Thus, $h = H \cdot \sigma_{\bar{x}} = 2.096 \times \frac{10}{\sqrt{9}} = 6.99$.

(b) Now, $\mu_{\bar{x}} = 110$ and $\sigma_{\bar{x}} = \frac{9}{\sqrt{9}} = 3$. Thus, $H^* = \frac{h}{\sigma_{\bar{x}}} = \frac{6.99}{3} = 2.33$,

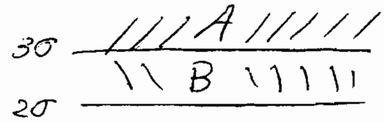
$$D^* = \frac{\left|\mu_{\bar{x}} - \frac{k_1+k_2}{2}\right|}{\sigma_{\bar{x}}} = \frac{\left|110 - \frac{104+96}{2}\right|}{3} = 3.33, \text{ and } K^* = \frac{k_1-k_2}{2\sigma_{\bar{x}}} = \frac{104-96}{6} = 1.33.$$

From Table A.5, we have $ARL < 4$ (rough estimate).

(Another way is to note from part (a) that $ARL \doteq 2.5$ for $\delta = 8$ and $\sigma = 10$ and now $\delta = 10$ and $\sigma = 9$. Thus, $ARL < 2.5$.)

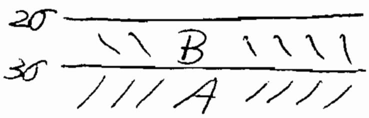
4. Let A be the region outside 3 sigma control limits and B be the region outside 2 sigma limits but within 3 sigma limits.

Let $P_A = P(Q_i \in A)$, $P_B = P(Q_i \in B)$



and S_1 = "all OK"

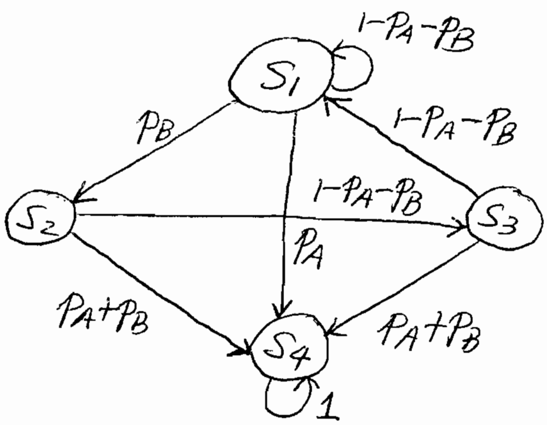
S_2 = "no alarm yet and current point is in region B"



S_3 = "no alarm yet and the previous point is in region B"

S_4 = "alarm".

Then we have the following MC diagram:



The transition matrix is

$$P_{4\times4} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} & \\ \left[\begin{array}{ccc|c} 1-P_A-P_B & P_B & 0 & P_A \\ 0 & 0 & 1-P_A-P_B & P_A+P_B \\ 1-P_A-P_B & 0 & 0 & P_A+P_B \\ \hline 0 & 0 & 0 & 1 \end{array}\right] & \begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \end{array}$$

The desired ARL is the first element of $L = (I-R)^{-1} 1_{3\times 1}$, where

$$R = \begin{bmatrix} 1-P_A-P_B & P_B & 0 \\ 0 & 0 & 1-P_A-P_B \\ 1-P_A-P_B & 0 & 0 \end{bmatrix}.$$