Stat/IE 531 Final Exam Solutions May 5, 2005

1. (a) Here, $Q = \bar{x}$, $\mu_{\bar{x}} = 200$, $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.5}{\sqrt{9}} = 0.5$, and $\delta = 1$.

Thus, shift $= \frac{\delta}{\sigma_{\bar{x}}} = \frac{1}{0.5} = 2$. Then by Table 4.2 of V&J, $\lambda^{opt} = 0.49$. Thus by Table 4.3 of V&J, we have

$$K = \left(\frac{0.49-0.4}{0.5-0.4}\right)\cdot(2.53) + \left(\frac{0.5-0.49}{0.5-0.4}\right)(2.50) = 2.527.$$ Then

$$UCL_{EWMA} = \mu_{\bar{x}} + K\cdot\sigma_{\bar{x}}\sqrt{\frac{\lambda^{opt}}{2-\lambda^{opt}}}$$

$$= 200 + 2.527\cdot 0.5\cdot\sqrt{\frac{0.49}{2-0.49}} = 200.72$$

$$LCL_{EWMA} = 200 - 0.72 = 199.28.$$

Since $\bar{x} = 201$ is outside the control limits, it would trigger an alarm.

(b) Now $\mu_{\bar{x}} = 201$ and $\sigma_{\bar{x}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$. Thus,

$$\mathcal{D}^* = \frac{\left|\mu_{\bar{x}} - \frac{UCL_{EWMA}+LCL_{EWMA}}{2}\right|}{\sigma_{\bar{x}}} = \frac{|201-200|}{2/3} = 1.5,$$

and $$K^* = \frac{UCL_{EWMA} - LCL_{EWMA}}{2\sigma_{\bar{x}}}\sqrt{\frac{2-\lambda^{opt}}{\lambda^{opt}}}$$

$$= \frac{200.72-199.28}{2\cdot(2/3)}\sqrt{\frac{2-0.49}{0.49}} = 1.896.$$

By Table A.3, we have $ARL \simeq 2.6$.

2. (a) Note that $E(t) = T(t) - Y(t)$ and $\Delta E(t) = E(t) - E(t-1)$.

$Z(0)=1$, $Y(0)=Z(0)=1$. $E(0)=T(0)-Y(0)=2-1=1$. $E(-1)=2$.

$\Delta X(0)=0.6\cdot(1-2)+0.3\times 1=-0.3$.

$Z(1)=2$, $Y(1)=Z(1)+\Delta X(0)=2-0.3=1.7$. $E(1)=T(1)-Y(1)=2-1.7=0.3$.

In general, we have the following.

t	$Z(t)$	$Y(t)$	$E(t)$	$\Delta X(t)=0.9E(t)-0.6E(t-1)$
0	1	1	1	-0.3
1	2	1.7	0.3	-0.33
2	3	2.37	-0.37	-0.513
3	4	2.857	2.143	2.1507
4	5	6.0077	-1.0077	-2.1927
5	6	4.815	0.185	0.7711

(b) Since $Y(t)=Z(t)+\sum_{s=0}^{t-1}\Delta X(s)=T(t)$ (for $t\geq 1$), we have $\Delta X(0)=T(1)-Z(1)=2-2=0$. In general, for $t\geq 1$, $\Delta X(t)=(T(t+1)-Z(t+1))-(T(t)-Z(t))$

$$=(T(t+1)-T(t))-(Z(t+1)-Z(t))=T(t+1)-T(t)-1.$$

That is, $\Delta X(1)=-1$, $\Delta X(2)=2$, $\Delta X(t)=-1$ for $t\geq 3$.

3. Let s_{ij}^2 be the sample variance at level i of A ("day") and level j of B within A ("heat (day)"), $i, j = 1, 2$.

That is, $s_{ij}^2 = \frac{1}{n_{ij}-1} \sum_{k=1}^{n_{ij}} (y_{ijk} - \bar{y}_{ij})^2$, where $n_{11} = n_{12} = n_{21} = 3$, and $n_{22} = 4$. Then, $\hat{\sigma}^2 = \frac{2(s_{11}^2 + s_{12}^2 + s_{21}^2) + 3 \cdot s_{22}^2}{9}$.

Let s_i^2 be the sample variance of $\bar{y}_{ij}$'s. Then

$$ES_1^2 = \frac{1}{2}\left(\sigma_\beta^2 + \frac{\sigma^2}{3} + \sigma_\beta^2 + \frac{\sigma^2}{3}\right) = \sigma_\beta^2 + \frac{1}{3}\cdot\sigma^2$$

$$ES_2^2 = \frac{1}{2}\left(\sigma_\beta^2 + \frac{\sigma^2}{3} + \sigma_\beta^2 + \frac{\sigma^2}{4}\right) = \sigma_\beta^2 + \frac{7}{24}\sigma^2.$$

$\Rightarrow \rho \cdot s_1^2 + (1-\rho) s_2^2 - \left[\frac{1}{24}\rho + \frac{7}{24}\right]\hat{\sigma}^2$ is a sensible estimat of σ_β^2 (for any given $\rho \in [0,1]$), denoted by $\hat{\sigma}_\beta^2$.

To estimat σ_α^2, note that

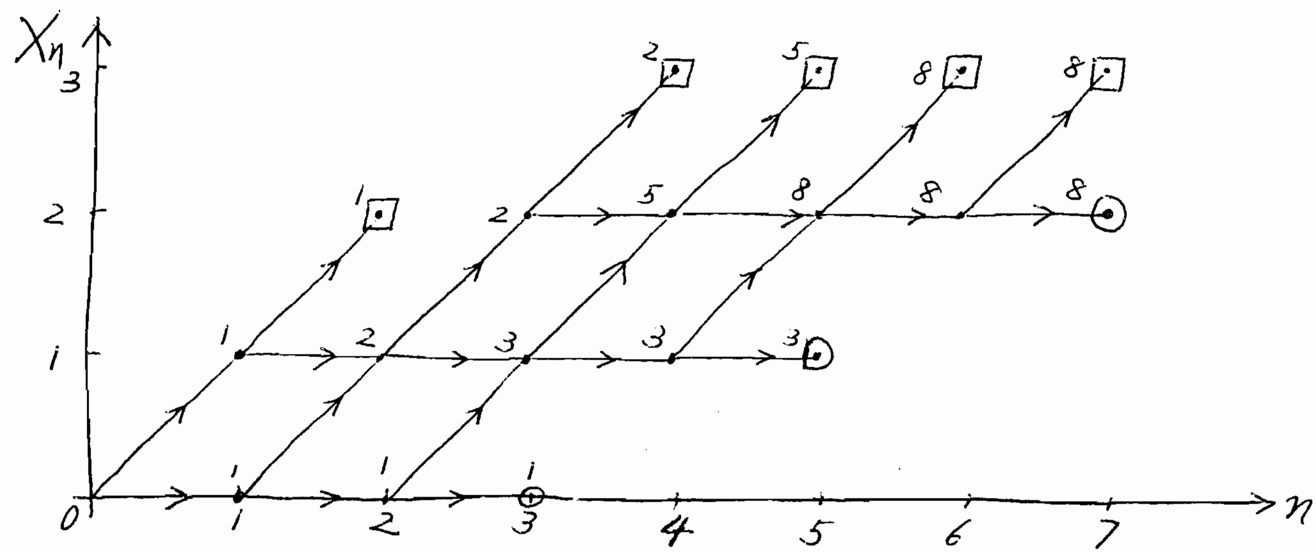
$$ES^2 = \frac{1}{2}\left(\sigma_\alpha^2 + \frac{1}{2}\sigma_\beta^2 + \frac{1}{6}\sigma^2 + \sigma_\alpha^2 + \frac{1}{2}\sigma_\beta^2 + \frac{7}{48}\sigma^2\right)$$

$$= \sigma_\alpha^2 + \frac{1}{2}\sigma_\beta^2 + \frac{5}{32}\sigma^2$$

$\Rightarrow s^2 - \frac{1}{2}\hat{\sigma}_\beta^2 - \frac{5}{32}\hat{\sigma}^2$ is a sensible estimator of σ_α^2.

(Note that s^2 is the sample variance of $\bar{y}_1$ and $\bar{y}_2$, that is, $s^2 = \frac{1}{2}(\bar{y}_1 - \bar{y}_2)^2$.)

4. Below is a diagram that gives all possible evolutions of (n, X_n).



□ reject ○ accept

(a) $P_a = \sum_{\substack{\text{acceptance} \\ \text{boundary}}} \begin{pmatrix} \text{path count from} \\ (0,0) \text{ to } (n, X_n) \end{pmatrix} \cdot p^{X_n} \cdot (1-p)^{n-X_n}$

$= 1 \cdot (1-p)^3 + 3 \cdot p(1-p)^4 + 8 \cdot p^2 \cdot (1-p)^5$

(Consider type B calculations throughout.)

$ASN = \sum_{\substack{\text{stopping} \\ \text{boundary}}} n \cdot \Pr[\text{ending at } (n, X_n)]$

$= 2p^2 + 3 \cdot (1-p)^3 + 4 \times 2 \cdot p^3(1-p) + 5 \times 3 p(1-p)^4 + 5 \times 5 \cdot p^3(1-p)^2$

$+ 6 \times 8 p^3(1-p)^3 + 7 \times 8 \cdot p^2(1-p)^5 + 7 \times 8 \cdot p^3(1-p)^4.$

(b) $$AOQ = \frac{1}{N} \sum_{\substack{\text{acceptance} \\ \text{boundary}}} (N-n)\cdot p \cdot P_r[\text{ending at } (n, X_n)]$$

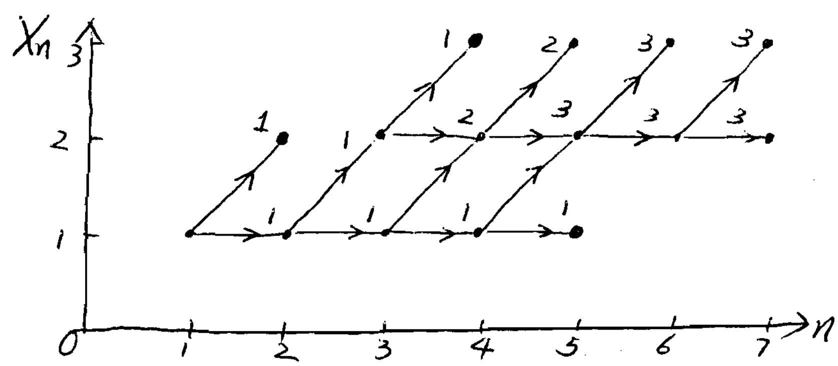
$$= \frac{1}{100}\left[97\cdot p\cdot(1-p)^3 + 95\cdot p\cdot 3\cdot p(1-p)^4 + 93\cdot p\cdot 8\cdot p^2(1-p)^5\right]$$

$$= \frac{p(1-p)^3}{100}\cdot\left[97 + 285\,p(1-p) + 744\,p^2(1-p)^2\right].$$

$$ATI = \sum_{\substack{\text{acceptance} \\ \text{boundary}}} n\cdot P_r[\text{ending at } (n, X_n)] + N\cdot(1-P_a)$$

$$= 3\cdot(1-p)^3 + 15\,p(1-p)^4 + 56\,p^2(1-p)^5 + 100\cdot(1-P_a)$$

(c) To obtain the UMVUE of p, we first obtain path count from $(1,1)$ to (n, X_n) as shown below.



Stop sampling point (n, X_n)	$(2,2)$	$(3,0)$	$(4,3)$	$(5,1)$	$(5,3)$	$(6,3)$	$(7,2)$	$(7,3)$
$\tilde{p}$ = UMVUE of p $= \dfrac{\text{path count from } (1,1) \text{ to } (n, X_n)}{\text{path count from } (0,0) \text{ to } (n, X_n)}$	1	0	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$

5. (a) Here, $L = 1.14$, $U = 1.16$, $\sigma = 0.003$, and $U - L = 0.02 > 6\sigma = 0.018$.

Thus, (8.38) and (8.39) of V&J can be used.

$$n \approx \left(\frac{Q_z(P_{a_1}) - Q_z(P_{a_2})}{Q_z(1-P_1) - Q_z(1-P_2)} \right)^2 = \left(\frac{Q_z(0.9) - Q_z(0.1)}{Q_z(0.95) - Q_z(0.9)} \right)^2$$

$$= \left(\frac{1.28 - (-1.28)}{1.645 - 1.28} \right)^2 = 49.2 \approx 49.$$

$$\Delta_2 \approx \sigma \left(\frac{Q_z(P_{a_1}) \cdot Q_z(1-P_2) - Q_z(P_{a_2}) \cdot Q_z(1-P_1)}{Q_z(P_{a_1}) - Q_z(P_{a_2})} \right)$$

$$= 0.003 \cdot \left(\frac{1.28 \times 1.28 - (-1.28) \times 1.645}{1.28 - (-1.28)} \right) = 4.388 \times 10^{-3}.$$

(b) Here, $\theta = \frac{P_1 + P_2}{2} = \frac{0.05 + 0.1}{2} = 0.075$, $\delta = P_2 - \theta = 0.025$.

Using (8.18) of V&J, we have

$$n \approx \left(Q_z(P_{a_2}) \right)^2 \cdot \left(\frac{(\theta + \delta)(1 - \theta - \delta)}{\delta^2} \right) = (-1.28)^2 \cdot \left(\frac{0.1 \times 0.9}{0.025^2} \right)$$

$= 235.9 \approx 236$, which is much larger than the sample size in part (a). (The acceptance number is $c \approx n \cdot \theta = 235.9 \times 0.075 = 17.7 \to 17$.)