Stat/IE 531 Midterm Exam Solutions 03/27/2003

1. (a) Sample #1 has range 0 in coded units. (The coded sample is $\{3, 3, 3, 3, 3\}$.) In coded units the confidence interval for μ is $(3-.5, 3+.5)$ and the interval for σ is $(0, \lambda_0) = (0, .666)$, where $\lambda_0 = .666$ for $n=5$ and $\alpha = .05$ from Table 1.5 of the notes.

Thus, in original units the intervals for μ and σ are respectively $(.34725, .34735)$ and $(0, .0000666)$ inch.

(b) The combined sample has range 1 in coded units. (The coded sample is $\{3, 3, 3, 3, 3, 3, 4, 4, 3, 4\}$.)

(1) Using the notation of page 16 of the notes, we have $n=10$, $x^*=3$, $n_{x^*}=7$, $n_{x^*+1}=3$, $m=7$. For $n=10$ and $m=7$, Table 1.3 gives $\Delta_1 = .515$, $\Delta_2 = .129$. Then $\Delta_L = .515$, $\Delta_U = .129$.

Thus in coded units the interval is

$(3.5-.515,\ 3.5+.129)$, or $(2.985,\ 3.629)$.

In original units, this is

$(.3472985,\ .3473629)$ inch.

(2) The interval for σ is $(0, \Lambda_1 m)$ in coded units, that is $(0, .747)$ from Table 1.6.

So the interval in original units is

$(0, .0000747)$ inch.

2. (a) $k_1 = \mu_Q + \frac{\delta}{2} = 10 + \frac{.04}{2} = 10.02 \quad (Q = \bar{x})$.

So $\mathcal{K} = \dfrac{k_1 - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \dfrac{10.02 - 10}{.04/\sqrt{4}} = 1. \quad \left(\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\right)$

By Table 4.5, we have $\mathcal{H} = 2.18$ and thus

$h_1 = \mathcal{H} \cdot \sigma_{\bar{x}} = 2.18 \cdot \frac{.04}{\sqrt{4}} = 0.0436,$

with $u_0 = 0$.

(b) For the CUSUM scheme in part (a), using (4.15) and (4.16) of V&J, we have

$$\mathcal{H}^* = \frac{h_1}{\sigma_{\bar{x}}} = \frac{0.0436}{0.04/\sqrt{4}} = 2.18$$

$$\mathcal{S}^* = \frac{\mu_{\bar{x}} - k_1}{\sigma_{\bar{x}}} = \frac{10.03 - 10.02}{0.04/\sqrt{4}} = 0.5$$

Thus the ARL is around 5 from Table A.4.

(c) With an all-OK ARL of 370, the Shewhart $\bar{x}$ chart has 3 sigma control limits, which are

$10 \pm 3 \cdot \frac{.04}{\sqrt{4}}$, or $(9.94, 10.06)$. Thus

$P[\bar{x}$ plots outside the Shewhart limits$]$

$$= P[\bar{x} < 9.94] + P[\bar{x} > 10.06] \qquad (\mu_{\bar{x}} = 10.03, \sigma_{\bar{x}} = 0.02)$$

$$= P\left[Z < \frac{9.94 - 10.03}{0.02}\right] + P\left[Z > \frac{10.06 - 10.03}{0.02}\right]$$

$$= P[Z < -4.5] + P[Z > 1.5] \approx .000 + .0668 = .0668$$

So the ARL for the Shewhart chart is about

$$\frac{1}{.0668} \approx 15.$$

3. (a) (1) $\tilde{\sigma} = \dfrac{\bar{R}}{d_2(n)} = \dfrac{(74/18)}{2.326} \approx 1.767$ $(n=5)$.

(2) $\hat{\sigma} = \dfrac{\bar{S}}{c_4(n)} = \dfrac{(31.51/18)}{0.9400} \approx 1.862$

(b) The control limits for S_i are

$$UCL = B_6 \cdot \tilde{\sigma} = (1.964)\cdot(1.767) \approx 3.47$$

$$LCL = B_5 \cdot \tilde{\sigma} = 0 \cdot \tilde{\sigma} = 0.$$

(c) The control limits for $\bar{x}$'s are

$$UCL = \bar{\bar{x}} + A_2 \cdot \bar{R} = \frac{200.4}{18} + (0.577)\cdot\frac{74}{18} \approx 13.505$$

$$LCL = \bar{x} - A_2 \cdot \bar{R} = \frac{200.4}{18} - (0.577)\cdot\frac{74}{18} \approx 8.761$$

(d) The threading process appears to be stable based on the control limits in part (c) and the $\bar{x}$'s values since these $\bar{x}$ values all fall within the limits.

4 (a) Here, the observed variation in the measurements involves both "between operator" variation and "within operator" variation. Thus the standard deviation of interest is

$$\sigma_{overall} = \sqrt{\sigma^2_{meas.} + \sigma^2_{reproducibility}} = \sqrt{\sigma^2 + \sigma^2_\beta + \sigma^2_{\alpha\beta}}.$$

From page 27 of V&J, we have

$$\hat{\sigma}_{overall} = \sqrt{MSE + \frac{1}{mI}(MSB - MSAB) + \frac{1}{m}(MSAB - MSE)}$$

$$= \sqrt{\frac{1}{mI}\left[(m-1) I \cdot MSE + MSB + (I-1)\, MSAB\right]}$$

$$= \sqrt{\frac{1}{2\times 2}\cdot\left[1\cdot 2\cdot(2.0955) + .6730 + (2-1)\cdot 6.1245\right]}$$

$$\approx 1.657.$$

(b) By (1.3) of the notes, we have

$$\sqrt{\widehat{Var}(\hat{\sigma}_{overall})} = \sqrt{\frac{1}{2\,\hat{\sigma}^2_{overall}}\cdot\frac{1}{(mI)^2}\left[\frac{((m-1)\cdot I\cdot MSE)^2}{df_{MSE}} + \frac{(MSB)^2}{df_{MSB}} + \frac{((I-1)MSAB)^2}{df_{MSAB}}\right]}$$

$$= \sqrt{\frac{1}{2\cdot(2.747)}\cdot\frac{1}{4^2}\cdot\left[\frac{(2\times 2.0955)^2}{10} + \frac{0.673^2}{4} + \frac{6.1245^2}{4}\right]} \approx 0.3577.$$

<u>Derivation</u>: Let $c_1 = \frac{(m-1)\cdot I}{mI}$, $c_2 = \frac{1}{mI}$, $c_3 = \frac{I-1}{mI}$, and

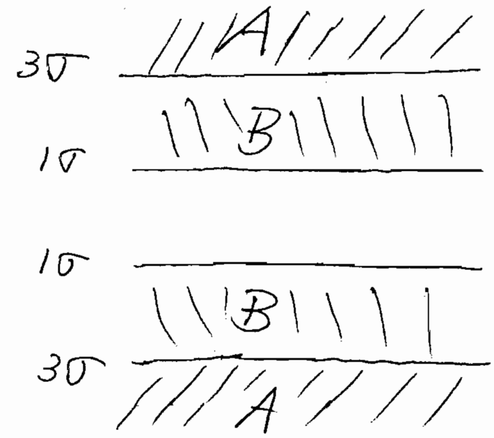
$g(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3$. Then

$\hat{\sigma}_{overall} = \sqrt{g(MSE, MSB, MSAB)}$ and by (1.3) of the notes, we have

$$\sqrt{\widehat{Var}(\hat{\sigma}_{overall})} = \sqrt{\frac{1}{2\hat{\sigma}^2_{overall}}\left[\frac{c_1^2 MSE^2}{df_{MSE}} + \frac{c_2^2 MSB^2}{df_{MSB}} + \frac{c_3^2 MSAB^2}{df_{MSAB}}\right]}$$

since $\left(\frac{\partial}{\partial x_i}\sqrt{g}\right)^2 = \left(\frac{c_i}{2\sqrt{g}}\right)^2 = \frac{c_i^2}{4\hat{\sigma}^2_{overall}}$ for $x_1 = MSE$, $x_2 = MSB$, $x_3 = MSAB$.

5. Let A be the region outside 3 sigma control limits and B be the region outside 1 sigma control limits but within 3 sigma limits.



Let $\delta_A = P(Q_i \in A)$,

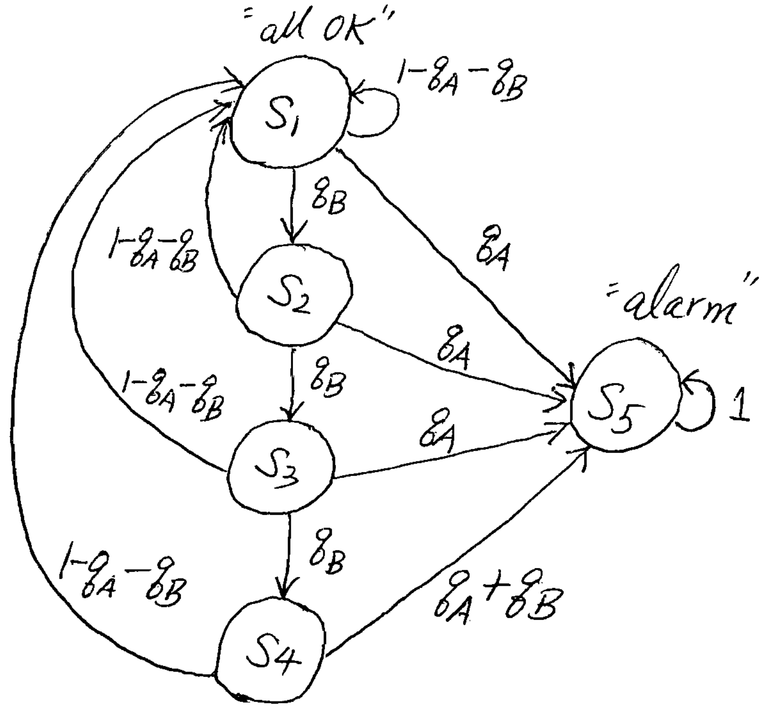
$\delta_B = P(Q_i \in B)$.

Let S_1 = "all OK",

S_{i+1} = "no alarm yet and i consecutive points are in region B", $i = 1, 2, 3$,

and S_5 = "alarm".

Then we have the following MC diagram:



The transition matrix is

$$\underset{5\times 5}{P} = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \end{array} \begin{array}{c} \begin{array}{ccccc} S_1 & S_2 & S_3 & S_4 & S_5 \end{array} \\ \left[\begin{array}{cccc:c} 1-g_A-g_B & g_B & 0 & 0 & g_A \\ 1-g_A-g_B & 0 & g_B & 0 & g_A \\ 1-g_A-g_B & 0 & 0 & g_B & g_A \\ 1-g_A-g_B & 0 & 0 & 0 & g_A+g_B \\ \hdashline 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

The desired ARL is the mean number of steps from S_1 to S_5, i.e., L_1, where L_1 is the first element of $L=(I-R)^{-1}\cdot 1$.