Stat/IE 531 Final Exam Solutions May 9, 2003

1. (a) $Q=\bar{x}$, $\mu_{\bar{x}} = 5$, $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{.2}{\sqrt{4}} = .1$.

$\delta = .2 \Rightarrow \text{shift} = \frac{\delta}{\sigma_{\bar{x}}} = \frac{.2}{.1} = 2$. Then by Table 4.2 of V&J, $\lambda^{opt} = .36$. Thus by Table 4.3 of V&J,

$K = \left(\frac{.36-.3}{.4-.3}\right)\cdot(3.05) + \left(\frac{.4-.36}{.4-.3}\right)(3.02) = 3.038$. We then have

$$UCL_{EWMA} = \mu_{\bar{x}} + K\cdot\sigma_{\bar{x}}\cdot\sqrt{\frac{\lambda^{opt}}{2-\lambda^{opt}}}$$

$$= 5 + (3.038)\cdot(.1)\cdot\sqrt{\frac{.36}{2-.36}} = 5.142$$

$$LCL_{EWMA} = \mu_{\bar{x}} - K\cdot\sigma_{\bar{x}}\cdot\sqrt{\frac{\lambda^{opt}}{2-\lambda^{opt}}} = 4.858.$$

(b) Now $\mu_{\bar{x}} = 5.3$ and $\sigma_{\bar{x}} = \frac{.3}{\sqrt{4}} = .15$. Thus

$$D^* = \frac{\left|\mu_{\bar{x}} - \frac{UCL_{EWMA}+LCL_{EWMA}}{2}\right|}{\sigma_{\bar{x}}} = \frac{|5.3-5|}{.15} = 2,$$

and $K^* = \frac{5.142-4.858}{2\times 0.15}\sqrt{\frac{2-.36}{.36}} = 2.02$.

By Table A.3. we have $ARL \approx 2.1$.

2. (a) Since $\alpha+\beta t$ is a constant for given t, the

$Z(t)$'s are independent. Thus, for $s>t$, $\hat{Z}(s|t)=$

$E_{\mathcal{F}}(Z(s)|\ldots, Z(-1), Z(0), \cdots, Z(t)) = E_{\mathcal{F}}(Z(s)) = \alpha+\beta s$.

(b) Note that $A(a(t), 1) = T(t+1) - \{\hat{Z}(t+1|t) + \sum_{s=0}^{t-1} A(a(s), t+1-s)\}$

or $(1-2^{-1})\cdot a(t) = -(\alpha+\beta(t+1) + \sum_{s=0}^{t-1}(1-2^{-(t+1-s)})\cdot a(s))$.

Thus $\frac{1}{2}\cdot a(0) = -(\alpha+\beta)$, or $a(0) = -2(\alpha+\beta)$.

$$(1-2^{-1})a(1) = -(\alpha+2\beta+(1-2^{-2})\cdot a(0))$$

$$\Rightarrow a(1) = -(\beta-\alpha) = \alpha-\beta.$$

$$(1-2^{-1})a(2) = -(\alpha+3\beta+(1-2^{-3})a(0)+(1-2^{-2})a(1))$$

$$\Rightarrow a(2) = -\beta.$$

(c) $(1-2^{-1})a(3) = -(\alpha+4\beta+(1-2^{-4})a(0)+(1-2^{-3})a(1)$

$+(1-2^{-2})a(2))$

$$\Rightarrow a(3) = -\beta.$$

In general, by mathematical induction, if $a(r) = -\beta$ for $2\le r\le t$, then it is easy to verify that $a(t+1) = -\beta$. Thus $a(t) = -\beta$ for all $t\ge 2$.

So clearly $a(t)$ has a limit $-\beta$ because $a(t)$ is a constant $-\beta$ for all $t \geq 2$.

(d) Note that
$$Y(t) = Z(t) + \sum_{s=0}^{t-1} A(a(s), t-s)$$
$$= Z(t) + \sum_{s=0}^{t-1} (1-2^{-(t-s)}) \cdot a(s).$$

Thus
$$Y(1) = Z(1) + (1-2^{-1})\,a(0)$$
$$= \alpha+\beta+\varepsilon(1) + (1-2^{-1})(-2)(\alpha+\beta) = \varepsilon(1).$$

Similarly, for $t \geq 2$, we have
$$\sum_{s=0}^{t-1} (1-2^{-(t-s)})\,a(s) = (1-2^{-t})\,a(0) + (1-2^{-(t-1)})\,a(1)$$
$$+ \sum_{s=2}^{t-1} (1-2^{-(t-s)})\,a(s)$$
$$= (1-2^{-t})(-2\alpha-2\beta) + (1-2^{-(t-1)}) \cdot (\alpha-\beta) - \sum_{s=2}^{t-1} (1-2^{-(t-s)})\,\beta$$
$$= -\alpha-(t+1)\beta.$$

Thus, $Y(t) = \varepsilon(t)$ for all $t \geq 1$.

Note that $EY(t) = 0$ and $Var(Y(t)) = \sigma^2$ while $EZ(t) = \alpha+\beta t$ and $Var(Z(t)) = \sigma^2$. Thus, the controlled process is on target while the uncontrolled process drifts.

3. (a) Note that $EMSC(B(A)) = \sigma^2$

$$EMSB(A) = \sigma^2 + K\sigma_\beta^2$$

$$EMSA = \sigma^2 + K\sigma_\beta^2 + JK\sigma_\alpha^2.$$

Thus, sensible estimates of σ^2, σ_α^2 and σ_β^2 are

$$\hat{\sigma}^2 = MSC(B(A)) = MSE = .0256.$$

$$\hat{\sigma}_\beta^2 = \max\left(0, \frac{MSB(A) - MSC(B(A))}{K}\right) = 0$$

and $\hat{\sigma}_\alpha^2 = \max\left(0, \dfrac{MSA - MSB(A)}{JK}\right) = .00283.$

(Here, $I=2$, $J=2$, $K=3$, $MSA = .0184$, $MSB(A) = .0014$, and $MSC(B(A)) = .0256$.)

(b) Note that $\hat{\sigma}_\alpha^2 = \frac{1}{JK} \cdot MSA + \left(-\frac{1}{JK}\right) MSB(A)$

is a linear function of MSA and $MSB(A)$.

Thus by the formulas on page 7 of the notes, a standard error of $\hat{\sigma}_\alpha^2$ is given by

$$\sqrt{\hat{Var}(\hat{\sigma}_\alpha^2)} = \sqrt{2\left[\left(\frac{1}{JK}\right)^2 \cdot \frac{(MSA)^2}{df_{MSA}} + \left(\frac{-1}{JK}\right)^2 \cdot \frac{(MSB(A))^2}{df_{MSB(A)}}\right]}$$

$$= \sqrt{2 \cdot \frac{1}{6^2} \cdot \left(\frac{.0184^2}{1} + \frac{.0014^2}{2} \right)} = .00434.$$

(c) The expected value of the overall sample variance is

$$ES^2 = \frac{1}{IJK-1} E \sum (y_{ijk} - \bar{y})^2 = \frac{1}{IJK-1} \cdot ESS_{Total}$$

$$= \frac{1}{IJK-1} \{ ESSA + ESSB(A) + ESSC(B(A)) \}$$

$$= \frac{1}{IJK-1} \cdot \{ (I-1)(KJ\sigma_\alpha^2 + K\sigma_\beta^2 + \sigma^2) + I(J-1)(K\sigma_\beta^2 + \sigma^2) + IJ(K-1)\sigma^2 \}$$

$$= \frac{1}{50-1} \{ 4 \cdot (10\sigma_\alpha^2 + \sigma_\beta^2 + \sigma^2) + 45(\sigma_\beta^2 + \sigma^2) + 0 \}$$

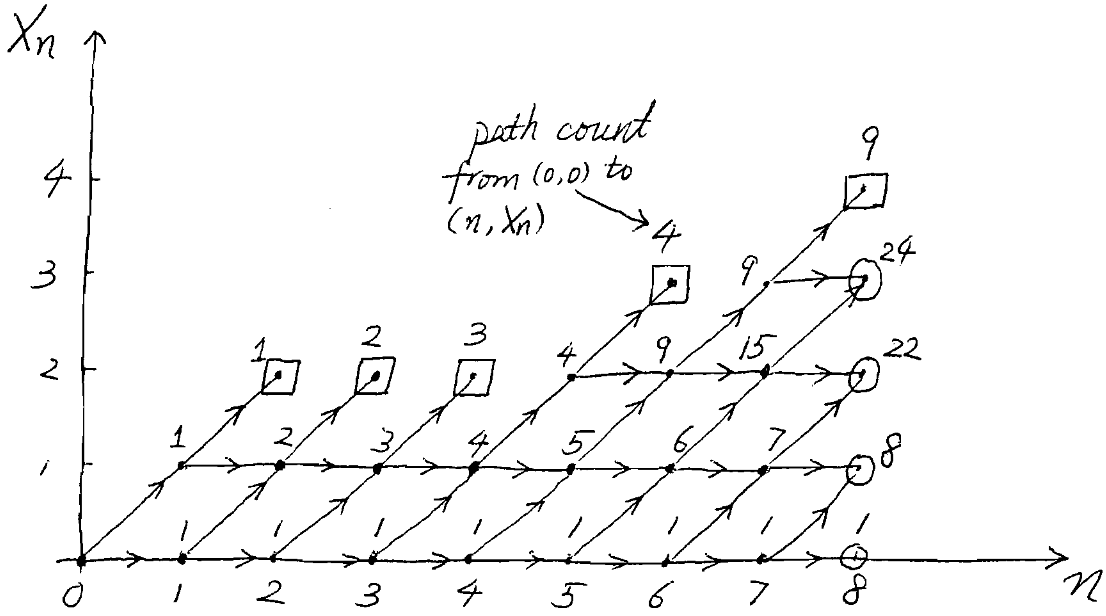
$$(I=5,\ J=10,\ K=1)$$

$$= \frac{1}{49} (40\sigma_\alpha^2 + 49\sigma_\beta^2 + 49\sigma^2).$$

So a prediction of the overall sample variance is

$$\frac{40}{49} \hat{\sigma}_\alpha^2 + \hat{\sigma}_\beta^2 + \hat{\sigma}^2 = \frac{40}{49} \cdot (.0028) + 0 + .0256 \approx 0.0279.$$

4. We first need to draw a diagram that gives all possible evolutions of (n, X_n). Note that (a maximum of) 8 items are inspected.



□ reject ○ accept

$$\text{(a)}\quad P_a = \sum_{\substack{\text{acceptance}\\ \text{boundary}}} \begin{pmatrix}\text{path count}\\ \text{from } (0,0)\\ \text{to } (n, X_n)\end{pmatrix} \cdot p^{X_n}(1-p)^{n-X_n}$$

$$= 24\cdot p^3(1-p)^5 + 22\cdot p^2(1-p)^6 + 8p(1-p)^7 + (1-p)^8$$

(Consider type B calculations throughout.)

$$ASN = \sum_{\substack{\text{stopping}\\ \text{boundary}}} n \cdot P[\text{ending at } (n, X_n)]$$

$$= (2)(1)p^2 + (3)(2)\cdot p^2(1-p) + (4)\cdot(3)\cdot p^2(1-p)^2$$
$$+ (6)(4)\cdot p^3(1-p)^3 + (8)(9)\cdot p^4(1-p)^4 + 8\cdot P_a,$$

where P_a is given above.

$$\text{(b)}\quad AOQ = \frac{1}{N}\cdot \sum_{\substack{\text{acceptance}\\ \text{boundary}}} (N-n)\cdot p \cdot P_r[\text{ending at } (n, X_n)]$$

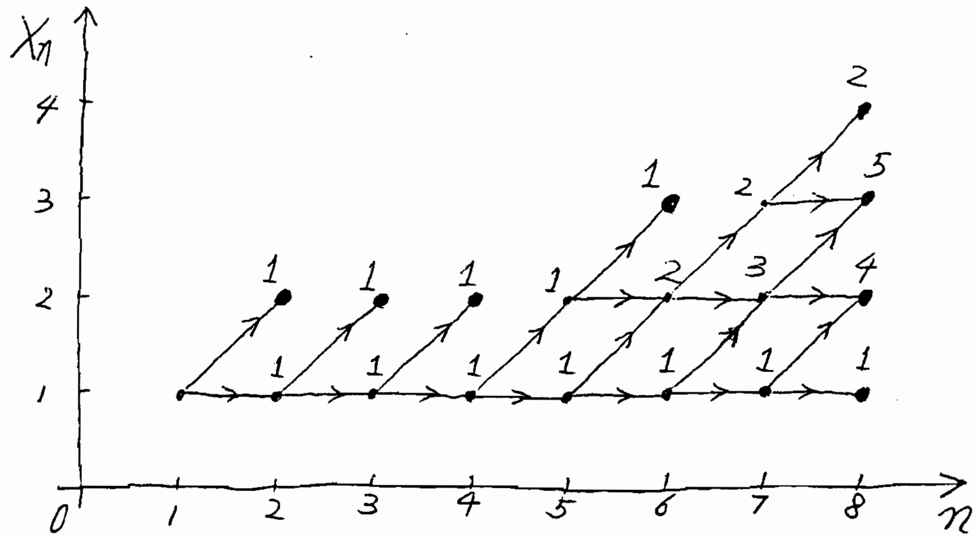
$$= \left(1-\frac{8}{50}\right)\cdot p \cdot P_a = \frac{21}{25}\cdot p \cdot P_a,$$

$$ATI = \sum_{\substack{\text{acceptance}\\ \text{boundary}}} n \cdot Pr[\text{ending at } (n, X_n)] + N\cdot(1-P_a)$$

$$= n\cdot P_a + N\cdot(1-P_a) = 50 - 42\cdot P_a,$$

where P_a is given in part (a).

(c) To obtain the UMVUE of p, we first obtain path count from $(1,1)$ to (n, X_n) as shown below.



Stop sampling point (n, X_n)	$\tilde{p}$ = UMVUE = $\dfrac{\text{path count from } (1,1) \text{ to } (n,X_n)}{\text{path count from } (0,0) \text{ to } (n,X_n)}$
(2, 2)	1
(3, 2)	1/2
(4, 2)	1/3
(6, 3)	1/4
(8, 0)	0
(8, 1)	1/8
(8, 2)	4/22
(8, 3)	5/24
(8, 4)	2/9

5. Note that $N=2$, $k_1=1$, $k_2=5$, $X/p \sim Bin(n,p)$.

$p \sim G$ with pmf $g(p) = 1/3$ for $p = .15, .2,$ or $.25$.

Then the posterior of p given x is

$$f(p|x) = \frac{f(x|p)\cdot g(p)}{f(x|p=.15)\cdot g(.15) + f(x|p=.2)\cdot g(.2) + f(x|p=.25)\cdot g(.25)}$$

$$= \frac{p^x (1-p)^{n-x}}{(.15)^x(.85)^{n-x} + (.2)^x(.8)^{n-x} + (.25)^x(.75)^{n-x}}, \quad p = .15, .2, .25.$$

① If $n=0$, then $c=0$ and $E(\text{total cost}) = \sum g(p)\cdot E(\text{total cost}|p)$

$= \sum g(p)\cdot N\cdot p\cdot k_2 = N\cdot k_2 \sum g(p)\cdot p = 2\times 5\times (\frac{1}{3})\cdot(.15+.2+.25) = 2.$

② If $n=1, x=0$, then $f(p|x) = f(p|0) = \dfrac{1-p}{.85+.8+.75}$

$$= \begin{cases} 17/48 & \text{if } p=.15 \\ 16/48 & \text{if } p=.2 \\ 15/48 & \text{if } p=.25 \end{cases}$$

$E(p|0) = \frac{17}{48}\cdot(.15) + \frac{16}{48}(.2) + \frac{15}{48}(.25) = \frac{19}{96}.$

If $n=1, x=1$, similarly, we have $f(p|1) = \begin{cases} \frac{1}{4} & \text{if } p=.15 \\ \frac{1}{3} & \text{if } p=.2 \\ \frac{5}{12} & \text{if } p=.25 \end{cases}$

Then $E(p|1) = \frac{5}{24}$. Since $E(p|x=0) < k_1/k_2 < E(p|x=1)$,

$c_G^{opt}(1) = 0.$

③ For $n=1$, $c=0$, $E(\text{total cost}) = P(X=0)\cdot E(\text{total cost}|X=0)$

$+ P(X=1)\cdot E(\text{total cost} \mid X=1).$

Note that $P(X=1) = \sum_i P(X=1 \mid P)\cdot g(P) = \sum_i P\cdot g(P)$

$= \frac{1}{3}(.15 + .2 + .25) = .2.$

Thus $P(X=0) = .8.$

$E(\text{total cost} \mid X=0) = n\cdot k_1 + (N-n)\cdot E(P \mid X=0)\cdot k_2$

$= 1\times 1 + (2-1)\times \frac{19}{96}\times 5$ $\quad (E(P \mid X=0) = \frac{19}{96}$ from ②$)$

$= 1 + \frac{95}{96} = \frac{191}{96}.$

$E(\text{total cost} \mid X=1) = N\cdot k_1 = 2.$

Thus, $E(\text{total cost}) = (.8)\cdot(\frac{191}{96}) + (.2)\cdot(2) = 1.992.$

④ If $n=2$, then $c=0$ and $E(\text{total cost}) = N\cdot k_1 = 2.$

⑤ Thus, the best inspection plan is $(n, c) = (1, 0).$