Stat/IE 531 Solutions to Homework #4

1. (Problem 2.8 of the notes)

   (a) Note that $Y \sim N(5,1)$ and $P(Y \notin (2,8)) = P(Y < 2) + P(Y > 8) = P(Z < -3) + P(Z > 3) = .0027$, where $Z \sim N(0,1)$. Then $X \sim Bin(n = 100, p = .0027)$, and $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 0.0303$, from which we have ARL $= 1/0.0303 \approx 33$.

   (b) Note that $P(Y \notin (2.29, 7.71)) = P(Z < -2.71) \times 2 = 0.0067$.
   Then, $P(X \geq 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) = 0.0304$, from which we have ARL $= 1/0.0304 = 32.89 \approx 33$.

   (c) Suppose $Y \sim N(6,1)$.

   - For the scheme in part (a), $P(Y \notin (2,8)) = P(Z < -4) + P(Z > 2) = .02278$ and $P(X \geq 2) = .6675$, which yields ARL $= 1/.6675 = 1.498$.
   - For the scheme in part (b), $P(Y \notin (2.29, 7.71)) = P(Z < -3.71) + P(Z > 1.71) = 0.0437$ and $P(X \geq 3) = 0.8181$, which yields ARL $= 1/0.8181 = 1.2224$.

2. (Problem 2.10 of the notes)
   $X \sim N(\mu, .04^2)$.

   (a) We want a two-sided CUSUM scheme and an EWMA scheme for $Q = X$, with a target value $= .13$, all-OK ARL $= 370$, and quickest possible detection of a change in mean of size $\Delta = .02$.

   - CUSUM scheme

$$
\begin{aligned}
k_1^{\text{opt}} &= \text{target} + \Delta/2 = .13 + .02/2 = .14 \\
k_2^{\text{opt}} &= \text{target} - \Delta/2 = .13 - .02/2 = .12 \\
\mathcal{K} &= \frac{k_1 - \text{target}}{\sigma_Q} = \frac{\text{target} - k_2}{\sigma_Q} = \frac{.14 - .13}{.04} = .25
\end{aligned}
$$

   From Table 4.6, we have $\mathcal{H} = 8.01$ and thus $h = h_1 = h_2 = \sigma_Q \mathcal{H} = .04(8.01) = .3204$.

   - EWMA scheme
   Shift $= \Delta/\sigma_Q = .02/.04 = .5$. From Tables 4.2 and 4.3, $\lambda^{\text{opt}} = .05$ and $\mathcal{K} = 2.49$.

$$
\begin{aligned}
UCL_{EWMA} &= \text{target} + \mathcal{K}\sigma_Q\sqrt{\lambda/(2-\lambda)} \\
&= .13 + 2.49(.04)\sqrt{.05/(2-.05)} \\
&= .13 + .0159 = .1459 \\
LCL_{EWMA} &= \text{target} - \mathcal{K}\sigma_Q\sqrt{\lambda/(2-\lambda)} \\
&= .13 - .0159 = .1141
\end{aligned}
$$

(b) For Shewhart,

$$\begin{aligned} UCL &= .13 + 3(.04) = .25 \\ LCL &= .13 - 3(.04) = .01 \end{aligned}$$

- $\mu_Q = .13$
  $P(X < .01 \text{ or } X > .25) = P(Z < -3) + P(Z > 3) = .0027$
  $ARL = 1/.0027 = 370.4 \qquad \ln ARL = 5.91$
- $\mu_Q = .12$ or $.14$
  $P(X < .01 \text{ or } X > .25) = P(Z > 2.75) + P(Z > 3.25) = .0036$
  $ARL = 1/.0036 = 281.2 \qquad \ln ARL = 5.64$
- $\mu_Q = .11$ or $.15$
  $P(X < .01 \text{ or } X > .25) = P(Z > 2.5) + P(Z > 3.5) = .0064$
  $ARL = 1/.0064 = 155.2 \qquad \ln ARL = 5.04$
- $\mu_Q = .10$ or $.16$
  $P(X < .01 \text{ or } X > .25) = P(Z > 2.25) + P(Z > 3.75) = .0123$
  $ARL = 1/.0123 = 370.4 \qquad \ln ARL = 4.4$
- $\mu_Q = .09$ or $.17$
  $P(X < .01 \text{ or } X > .25) = P(Z > 2) + P(Z > 4) = .0228$
  $ARL = 1/.0228 = 43.89 \qquad \ln ARL = 3.78$
- $\mu_Q = .08$ or $.18$
  $P(X < .01 \text{ or } X > .25) = P(Z > 1.75) + P(Z > 4.25) = .0401$
  $ARL = 1/.0401 = 24.96 \qquad \ln ARL = 3.22$
- $\mu_Q = .07$ or $.19$
  $P(X < .01 \text{ or } X > .25) = P(Z > 1.5) + P(Z > 4.5) = .0668$
  $ARL = 1/.0668 = 14.97 \qquad \ln ARL = 2.71$

For CUSUM scheme,

$(k_1+k_2)/2 = .13, \qquad \mathcal{H}^* = h/6\sigma_Q = 8.01, \qquad \mathcal{K}^* = (k_1-k_2)/(2\sigma_Q) = .02/(2\times.04) = .25$

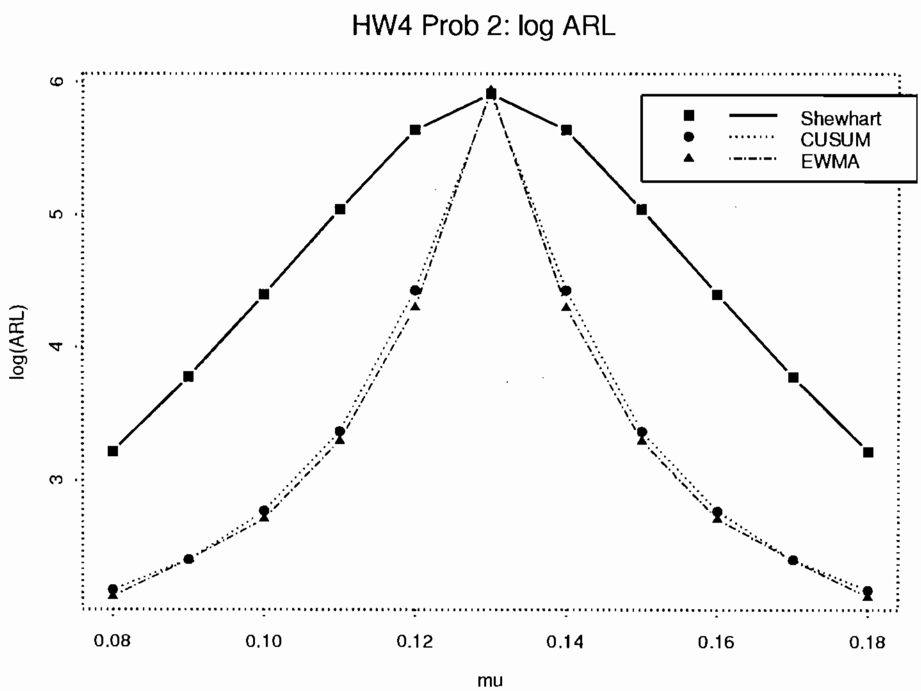
- $\mu_Q = .13$
  $D^* = (.13 - .13)/.04 = 0$, $ARL = 368$ (using Table A.5), $\ln ARL = 5.91$
- $\mu_Q = .12$ or $.14$
  $\mathcal{D}^* = (.14 - .13)/.04 = .25$, $ARL = 84$, $\ln ARL = 4.43$
- $\mu_Q = .11$ or $.15$
  $\mathcal{D}^* = (.15 - .13)/.04 = .5$, $ARL = 29$, $\ln ARL = 3.37$
- $\mu_Q = .10$ or $.16$
  $\mathcal{D}^* = (.16 - .13)/.04 = .75$, $ARL = 16$, $\ln ARL = 2.77$
- $\mu_Q = .09$ or $.17$
  $\mathcal{D}^* = (.17 - .13)/.04 = 1.0$, $ARL = 11$, $\ln ARL = 2.4$
- $\mu_Q = .08$ or $.18$
  $\mathcal{D}^* = (.18 - .13)/.04 = 1.25$, $ARL = 8.8$, $\ln ARL = 2.17$
- $\mu_Q = .07$ or $.19$
  $\mathcal{D}^* = (.19 - .13)/.04 = 1.50$, $ARL = 7.1$, $\ln ARL = 1.96$

For EWMA scheme,

$$EWMA_0 = \text{target} = .13, \qquad \sigma_Q = .04, \qquad UCL = .1459, \qquad LCL = .1141$$

$$\begin{aligned} \mathcal{K}^* &= (UCL - LCL)/(2\sigma_Q) \times \sqrt{(2-\lambda)/\lambda} \\ &= (.1459 - .1141)/(2 \times .04) \cdot \sqrt{(2 - .05)/.05} \\ &= .3975(6.245) = 2.4824 \approx 2.50 \end{aligned}$$

- $\mu_Q = .13$
  $D^* = (.13 - .13)/.04 = 0$, $ARL = 379$ (using Table A.3), $\ln ARL = 5.94$
- $\mu_Q = .12$ or $.14$
  $D^* = (.14 - .13)/.04 = .25$, $ARL = 74$, $\ln ARL = 4.3$
- $\mu_Q = .11$ or $.15$
  $D^* = (.15 - .13)/.04 = .5$, $ARL = 27$, $\ln ARL = 3.3$
- $\mu_Q = .10$ or $.16$
  $D^* = (.16 - .13)/.04 = .75$, $ARL = 15$, $\ln ARL = 2.71$
- $\mu_Q = .09$ or $.17$
  $D^* = (.17 - .13)/.04 = 1.0$, $ARL = 11$, $\ln ARL = 2.4$
- $\mu_Q = .08$ or $.18$
  $D^* = (.18 - .13)/.04 = 1.25$, $ARL = 8.3$, $\ln ARL = 2.12$

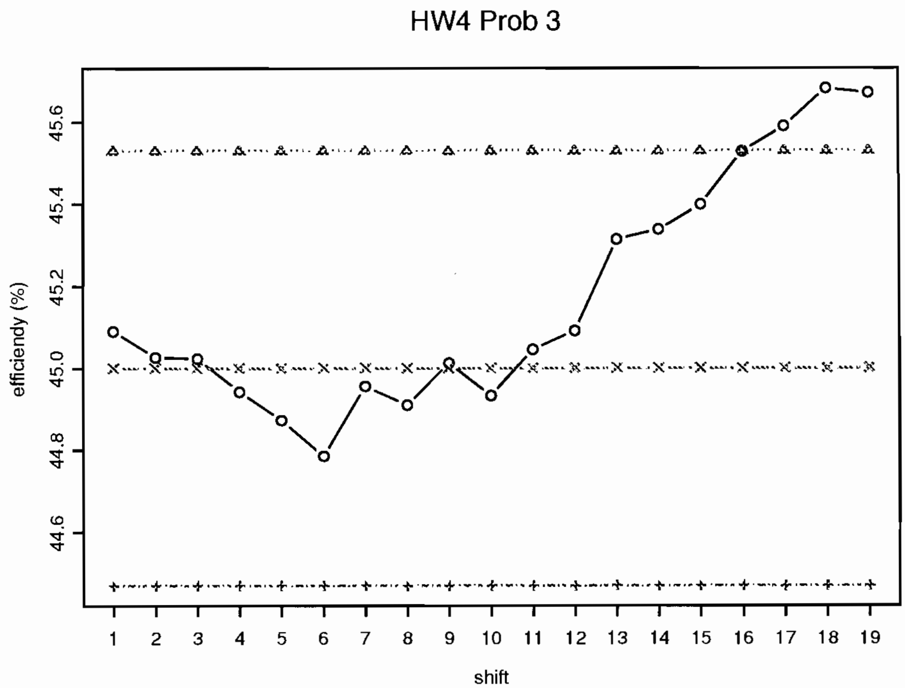


Clearly all three charts give the same on-target ARL, but the CUSUM and EWMA schemes yield much lower off-target ARL. There is really not much difference between the CUSUM and EWMA schemes.

3. (Problem 2.12 of the notes)

(e) $EWMA_0 = \text{target} = 45$, $\sigma_Q = .7$, $\delta = .7$, shift $= \delta/\sigma_Q = .7/.7 = 1$
By Table 4.2, $\lambda^{\text{opt}} = .13$ $(\delta = 1, ARL = 500)$
Using Table 4.3, we have $\mathcal{K} = 2.81 + \frac{.13-.10}{.20-.10}(2.96 - 2.81) = 2.855$. Therefore,
$UCL = \mu_Q + \mathcal{K}\sigma_Q\sqrt{\frac{\lambda}{2-\lambda}} = 45.53$ and $LCL = \mu_Q - \mathcal{K}\sigma_Q\sqrt{\frac{\lambda}{2-\lambda}} = 44.47$.

(f) Perhaps the mean efficiency shifted away from 45% at shift 13.



4. (Problem 3.2 of the notes)

(a)

t	Z(t)	T(t)	E(t)	DE(t)	DX(t)	Y(t)
-1	-3	0	3.0000000		0.00000000	-3.0000000
0	-3	0	3.0000000	0.00000000	1.50000000	-3.0000000
1	-3	0	1.5000000	-1.50000000	0.37500000	-1.5000000
2	-3	0	1.1250000	-0.37500000	0.46875000	-1.1250000
3	-3	0	0.6562500	-0.46875000	0.21093750	-0.6562500
4	-3	0	0.4453125	-0.21093750	0.16992188	-0.4453125
5	-3	0	0.2753906	-0.16992188	0.09521484	-0.2753906
6	-3	3	3.1801758	2.90478516	2.31628418	-0.1801758
7	-3	3	0.8638916	-2.31628418	-0.14712524	2.1361084
8	-3	3	1.0110168	0.14712524	0.54228973	1.9889832
9	-3	3	0.4687271	-0.54228973	0.09879112	2.5312729
10	-3	3	0.3699360	-0.09879112	0.16027021	2.6300640

(b)

t	Z(t)	T(t)	E(t)	DE(t)	DX(t)	Y(t)
-1	-3	0	3.0000000		0.00000000	-3.0000000
0	-3	0	3.0000000	0.00000000	1.50000000	-3.0000000
1	-3	0	1.5000000	-1.50000000	0.37500000	-1.5000000
2	-3	0	1.1250000	-0.37500000	0.46875000	-1.1250000
3	-3	0	0.6562500	-0.46875000	0.21093750	-0.6562500
4	-3	0	0.4453125	-0.21093750	0.16992188	-0.4453125
5	-3	0	0.2753906	-0.16992188	0.09521484	-0.2753906
6	3	3	-2.8198242	-3.09521484	-2.18371582	5.8198242
7	3	3	-0.6361084	2.18371582	0.22787476	3.6361084
8	3	3	-0.8639832	-0.22787476	-0.48896027	3.8639832
9	3	3	-0.3750229	0.48896027	-0.06527138	3.3750229
10	3	3	-0.3097515	0.06527138	-0.13855791	3.3097515

(c)

t	Z(t)	T(t)	E(t)	DE(t)	DX(t)	Y(t)
-1	-4	0	4.0000000		0.0000000	-4.000000
0	-3	0	3.0000000	-1.0000000	1.2500000	-3.000000
1	-2	0	0.7500000	-2.2500000	-0.1875000	-0.750000
2	-1	0	-0.0625000	-0.8125000	-0.2343750	0.062500
3	0	0	-0.8281250	-0.7656250	-0.6054688	0.828125
4	1	0	-1.2226562	-0.3945312	-0.7099609	1.222656
5	2	0	-1.5126953	-0.2900391	-0.8288574	1.512695
6	3	3	1.3161621	2.8288574	1.3652954	1.683838
7	4	3	-1.0491333	-2.3652954	-1.1158905	4.049133
8	5	3	-0.9332428	0.1158905	-0.4376488	3.933243
9	6	3	-1.4955940	-0.5623512	-0.8883848	4.495594
10	7	3	-1.6072092	-0.1116152	-0.8315084	4.607209

5. (Problem 3.4 of the notes)

(a)

t	$Z(t)$	$T(t)$	$Y(t)$	$E(t) = \Delta X(t)$
0	-1	0	-1	1
1	-1	0	0	0
2	-1	0	0	0
3	-1	0	0	0
4	-1	0	0	0
5	-1	0	0	0
6	-1	0	0	0
7	-1	0	0	0
8	-1	0	0	0
9	-1	0	0	0

(b)

t	$Z(t)$	$T(t)$	$Y(t)$	$E(t) = \Delta X(t)$
0	-1	0	-1	1
1	-1	0	-1	1
2	-1	0	0	0
3	-1	0	1	-1
4	-1	0	1	-1
5	-1	0	0	0
6	-1	0	-1	1
7	-1	0	-1	1
8	-1	0	0	0
9	-1	0	1	-1

6. (Problem 3.6 of the notes)

(c) Note that $A(a, s) = a$ for all $s \geq 1$ and

$$a(t) = -\frac{t+1}{t+2}Y(t) + \frac{t}{t+1}Y(t-1).$$

Then,

$$\begin{aligned}
Y(t) &= Z(t) + \sum_{s=0}^{t-1} A(a(s), t-s) = Z(t) + \sum_{s=0}^{t-1} a(s) \\
&= Z(t) + \sum_{s=0}^{t-1} \left[\frac{s}{s+1}Y(s-1) - \frac{s+1}{s+2}Y(s)\right] \\
&= Z(t) - \frac{t}{t+1}Y(t-1) \\
&= Z(t) - \frac{t}{t+1}\left[Z(t-1) - \frac{t-1}{t}Y(t-2)\right] \\
&= Z(t) - \frac{t}{t+1}\left[Z(t-1) - \frac{t-1}{t}\left\{Z(t-2) - \frac{t-2}{t-1}Y(t-3)\right\}\right]
\end{aligned}$$

$$
\begin{aligned}
&\vdots \\
&= Z(t) - \frac{1}{t+1}\Big[tZ(t-1) - (t-1)Z(t-2) + (t-2)Z(t-3) - \cdots + (-1)^{t-1}Z(0)\Big] \\
&= \epsilon(t-1) + \epsilon(t) - \frac{1}{t+1}[t\{\epsilon(t-2) + \epsilon(t-1)\} - (t-1)\{\epsilon(t-3) + \epsilon(t-2)\} \\
&\qquad + \cdots + (-1)^{t-1}\{\epsilon(-1) + \epsilon(0)\}\Big] \\
&= \epsilon(t) + \frac{1}{t+1}[\epsilon(t-1) - \epsilon(t-2) + \epsilon(t-3) - \cdots + (-1)^t\epsilon(0)]
\end{aligned}
$$

Since $\epsilon(t), \epsilon(t-1), \cdots$, are iid normal with mean 0 and variance σ^2, we have

$$
\begin{aligned}
E\{Y(t)\} &= 0 \\
\text{Var}\{Y(t)\} &= \sigma^2 + t\left(\frac{1}{t+1}\right)^2\sigma^2 = \left[1 + \frac{t}{(t+1)^2}\right]\sigma^2.
\end{aligned}
$$

Note that $E\{Z(t)\} = 0 = E\{Y(t)\}$ and $\text{Var}\{Z(t)\} = 2\sigma^2 > (1 + t/(t+1)^2)\sigma^2$, which suggests that the control algorithm stays on target and reduces the variances compared with the uncontrolled $Z(t)$.

(d)

$$
\begin{aligned}
\text{cov}(Y(t), Y(t-1)) &= \text{cov}\bigg(\epsilon(t) + \frac{1}{t+1}[\epsilon(t-1) - \epsilon(t-2) + \cdots], \\
&\qquad \epsilon(t-1) + \frac{1}{t}[\epsilon(t-2) - \epsilon(t-3) + \cdots]\bigg) \\
&= \frac{1}{t+1} - \frac{t-1}{t(t+1)} = \frac{1}{t(t+1)} \to 0 \qquad \text{as } t \to \infty
\end{aligned}
$$

This indicates that after some start-up periods it is reasonable to treat $Y(t)$ as iid normal $(0, \sigma^2)$ if all is OK.

7. (Problem 3.11 of the notes)
$Z(t) = \phi Z(t-1) + \epsilon(t) \qquad \Rightarrow \qquad E_{\mathcal{F}}[Z(t+1)|\cdots, Z(t)] \equiv \widehat{Z}(t+1|t) = \phi Z(t)$.
Assuming $A(a, s) = a$ for all $s \geq 1$, we have

$$
Y(t) = Z(t) + \sum_{s=0}^{t-1} A(a(s), t-s) = Z(t) + \sum_{s=0}^{t-1} a(s).
$$

(a) $(t = 0)$

Note that $\widehat{Z}(1|0) = \phi Z(0)$, and choose $a(0)$ so that $\widehat{Y}(1|0) = \widehat{Z}(1|0) + a(0) = T(1) = 0$.

Then, $a(0) = -\widehat{Z}(1|0) = -\phi Z(0)$.

$(t = 1)$

Note that $Y(1) = Z(1) + A(a(0), 1) = Z(1) - \phi Z(0)$, or $Z(1) = Y(1) + \phi Z(0)$. Then, we have

$$
\widehat{Z}(2|1) = \phi Z(1) = \phi\{Y(1) + \phi Z(0)\} = \phi Y(1) + \phi^2 Z(0).
$$

Choose $a(1)$ so that $\widehat{Y}(2|1) = \widehat{Z}(2|1) + a(0) + a(1) = 0$, then

$$
a(1) = -\widehat{Z}(2|1) - a(0) = -\phi Y(1) - \phi^2 Z(0) + \phi Z(0).
$$

$(t = 2)$

Note that $Y(2) = Z(2) + a(0) + a(1)$, or $Z(2) = Y(2) - a(0) - a(1)$.
Then we have

$$Z(3|2) = \phi Z(2) = \phi\{Y(2) - a(0) - a(1)\} = \phi Y(2) + \phi^2 Y(1) + \phi^3 Z(0)$$

Choose $a(2)$ so that $\widehat{Y}(3|2) = \widehat{Z}(3|2) + a(0) + a(1) + a(2) = 0$. Then

$$\begin{aligned} a(2) &= -\widehat{Z}(3|2) - a(0) - a(1) \\ &= -\phi\{Y(2) - a(0) - a(1)\} - a(0) - a(1) \\ &= -\phi Y(2) - (1-\phi)a(0) - (1-\phi)a(1) \\ &= -\phi Y(2) - (1-\phi)a(0) - (1-\phi)\{-\widehat{Z}(2|1) - a(0)\} \\ &= -\phi Y(2) + (1-\phi)\widehat{Z}(2|1) \\ &= -\phi Y(2) + (1-\phi)\{\phi Y(1) + \phi^2 Z(0)\} \end{aligned}$$

$(t = 3)$

Note that $Y(3) = Z(3) + a(0) + a(1) + a(2)$, or $Z(3) = Y(3) - a(0) - a(1) - a(2)$.
Then we have

$$\widehat{Z}(4|3) = \phi Z(3) = \phi\{Y(3) - a(0) - a(1) - a(2)\}.$$

Choose $a(3)$ so that $\widehat{Y}(4|3) = \widehat{Z}(4|3) + a(0) + a(1) + a(2) + a(3) = 0$. Then

$$\begin{aligned} a(3) &= -\widehat{Z}(4|3) - a(0) - a(1) - a(2) \\ &= -\phi\{Y(3) - a(0) - a(1) - a(2)\} - a(0) - a(1) - a(2) \\ &= -\phi Y(3) - (1-\phi)a(0) - (1-\phi)a(1) - (1-\phi)a(2) \\ &= -\phi Y(3) - (1-\phi)a(0) - (1-\phi)a(1) + (\phi - 1)\{-\widehat{Z}(3|2) - a(0) - a(1)\} \\ &= -\phi Y(3) + (1-\phi)\widehat{Z}(3|2) \\ &= -\phi Y(3) + (1-\phi)\{\phi Y(2) + \phi^2 Y(1) + \phi^3 Z(0)\}. \end{aligned}$$

In general,

$$\begin{aligned} a(t) &= -\phi Y(t) + (1-\phi)\widehat{Z}(t+1|t) \\ &= -\phi Y(t) + (1-\phi)\{\phi Y(t-1) + \phi^2 Y(t-2) + \cdots + \phi^{t-1} Y(1) + \phi^t Z(0)\} \end{aligned}$$

Since successive $a(\cdot)$ depend on all Y's in the past and $Z(0)$, this cannot be a PID control.

(b) Note that $Z(t+2|t) = E_{\mathcal{F}}[Z(t+2)| \cdots, Z(t)] = \phi^2 Z(t)$.

$(t = 0)$

Since $\widehat{Z}(2|0) = \phi^2 Z(0)$ and $a(0)$ is chosen so that $\widehat{Y}(2|0) = \widehat{Z}(2|0) + a(0) = 0$, we have $a(0) = -\widehat{Z}(2|0) = -\phi^2 Z(0)$.

$(t = 1)$

Note that $Y(1) = Z(1) + A(a(0), 1) = Z(1)$ since $A(a(0), 1) = 0$.

Then, $\widehat{Z}(3|1) = \phi^2 Z(1) = \phi^2 Y(1)$. Choosing $a(1)$ such that $\widehat{Y}(3|1) = \widehat{Z}(3|1) + A(a(0),3) + A(a(1),2) = \widehat{Z}(3|1) + a(0) + a(1) = 0$, we have

$$\begin{aligned} a(1) &= -\widehat{Z}(3|1) - a(0) \\ &= -\{\phi^2 Y(1)\} - \{-\phi^2 Z(0)\} \\ &= -\phi^2 Y(1) + \phi^2 Z(0) \end{aligned}$$

$(t = 2)$

Note that $Y(2) = Z(2) + A(a(0),2) + A(a(1),1) = Z(2) - \phi^2 Z(0)$ since $A(a(1),1) = 0$. Then, $\widehat{Z}(4|2) = \phi^2 Z(2) = \phi^2 Y(2) + \phi^4 Z(0)$. Choosing $a(2)$ such that $\widehat{Y}(4|2) = \widehat{Z}(4|2) + A(a(0),4) + A(a(1),3) + a(2) = 0$, we have

$$\begin{aligned} a(2) &= -\widehat{Z}(4|2) - a(0) - a(1) \\ &= -\{\phi^2 Y(2) + \phi^4 Z(0)\} - a(0) - \{-\widehat{Z}(3|1) - a(0)\} \\ &= -\{\phi^2 Y(2) + \phi^4 Z(0)\} + \widehat{Z}(3|1) \\ &= -\phi^2 Y(2) + \phi^2 Y(1) - \phi^4 Z(0) \end{aligned}$$

$(t = 3)$

Since $A(a(2),1) = 0$, we have

$$Y(3) = Z(3) + A(a(0),3) + A(a(1),2) + A(a(2),1) = Z(3) + a(0) + a(1).$$

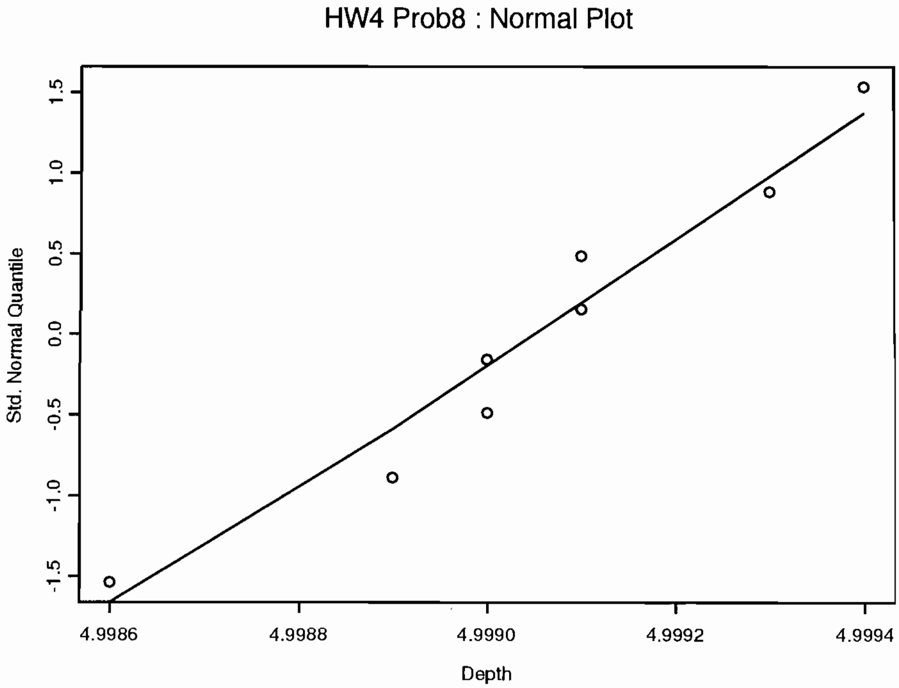
Then, $\widehat{Z}(5|3) = \phi^2 Z(3) = \phi^2\{Y(3) - a(0) - a(1)\}$.
Choosing $a(3)$ such that $\widehat{Y}(5|3) = \widehat{Z}(5|3) + a(0) + a(1) + a(2) + a(3) = 0$, we have

$$\begin{aligned} a(3) &= -\widehat{Z}(5|3) - a(0) - a(1) - a(2) \\ &= -\phi^2 Y(3) + (\phi^2 + 1)\{a(0) + a(1)\} - a(2) \\ &= -\phi^2 Y(3) + \phi^2 Y(2) - \phi^4 Y(1) + \phi^4 Z(0) \end{aligned}$$

Following this same framework we need to obtain $a(n)$ by computing $a(0), a(1), \cdots, a(n-1)$.
This is not a PID control since we eventually use all previous observations to decide a course of action.

8. (Problem 4.1 of the notes)

(a)



The slope of the simple linear regression is 3920.7 and an estimate of σ is $1/3920.7 = 0.000255$.

(b) A point estimate of 6σ is $6s = 6(0.000245) = 0.0014697$, and a 90% two-sided confidence interval for 6σ is given as

$$\left(6s\sqrt{\frac{n-1}{\chi^2_{.95,7}}}\ ,\ 6s\sqrt{\frac{n-1}{\chi^2_{.05,7}}} \right) = (0.00104, 0.00264).$$

(c) Upper and lower specifications are $U = 5.000$ and $L = 4.998$. Then a point estimate for C_p is
$\widehat{C}_p = \frac{U-L}{6s} = 1.3608$, and a 90% two-sided confidence interval for C_p is given as

$$\left(\frac{U-L}{6s}\sqrt{\frac{\chi^2_{.05,7}}{n-1}}\ ,\ \frac{U-L}{6s}\sqrt{\frac{\chi^2_{.95,7}}{n-1}} \right) = (0.75721, 1.92911)$$

(d) A point estimate for C_{pk} is

$$\widehat{C}_{pk} = \frac{U - L - 2\left|\bar{x} - \frac{U+L}{2}\right|}{6\sigma} = \frac{5 - 4.998 - 2|4.99905 - 4.999|}{.00147} = 1.29279.$$

Then a 95% lower confidence bound for C_{pk} is computed by

$$\widehat{C}_{pk} - z_{.95}\sqrt{\frac{1}{9n} + \frac{\widehat{C}^2_{pk}}{2n-2}} = 0.69227.$$

(e) A 95% two-sided prediction interval for the next depth measurement is given by

$$\bar{x} \pm (t_{.975,7}) \cdot s \cdot \sqrt{1 + \frac{1}{n}} = (4.99844, 4.99966).$$

(f) A 99% two-sided tolerance interval for 95% of all depth measurements produced by this process is given by $\bar{x} \pm t_2 s$ where $t_2 = 4.968$ from Table A.9a for 99% confidence and $p = .95$. This is computed as $4.99905 \pm 4.968(.000245) = (4.99783, 5.00027)$.

9. (Problem 4.3 of the notes)

$$
\begin{aligned}
\Pr\left[\Phi\left(\frac{\bar{x}+ks-\mu}{\sigma}\right) \geq .95\right] &= \Pr\left[\frac{\bar{x}+ks-\mu}{\sigma} \geq 1.645\right] \\
&= \Pr[\bar{x}+ks \geq 1.645\sigma+\mu] \\
&= \Pr\left[X \geq \frac{(1.645\sigma+\mu)-(\mu+k\sigma)}{\sqrt{\sigma^2\left(\frac{1}{n}+\frac{k^2}{2n}\right)}}\right] = .99 \\
\Rightarrow \quad & \frac{1.645-k}{\sqrt{\frac{1}{n}+\frac{k^2}{2n}}} = 2.33
\end{aligned}
$$

$$
1.645^2 - 2k(1.645) + k^2 = \frac{2.33^2}{26} + \frac{2.33^2k^2}{52}
$$

$$
.896k^2 - 3.29k + 2.49722 = 0 \qquad \Rightarrow \qquad k = \underline{2.5999} \text{ or } 1.072
$$

Compare this to the exact value obtained from Table A.9b, which is 2.638 for $n = 25$ and $p = .95$.