$\boxed{2.9}$ $\mu_Q = n \cdot p = (400)(0.025) = 10$

$\sigma_Q = \sqrt{npq} = 3.1225$

$\therefore$ Here,

$$\mathcal{H}^* = h_1/\sigma_Q \simeq 3.2$$

$$\delta^* = (\mu_Q - k_1)/\sigma_Q \simeq 0.64$$

Assume the normality of the binomial dsn. of Q.

Now, using table A.4 in the text, linear interpolation can be used to get ARL ≈ 6.

$\therefore$ An approximate ARL $\simeq 6$

$\boxed{2.11}$

(a). From table 4.7, for All-OK ARL = 370 and off-target ARL = 5, we get the value of $\sqrt{n}\,\delta/\sigma \geq 1.42$

Here, $\delta/\sigma = (800-750)/60 = 5/6$

$\Rightarrow n \geq 2.9$. So take $n = 3$

now $\mu_Q = 800$, $\sigma_Q = \sigma/\sqrt{n} = 34.65$

now, $k_2 = k^{opt} = \mu_Q - \delta/2 = 800 - 25 = 775$

$\mathcal{K} = (\mu_Q - k_2)/\sigma_Q = 0.72$ and All-OK ARL = 370, from table A.5 of text $\Rightarrow \mathcal{H} \approx 3.02$

$\Rightarrow h_2 = \mathcal{H} \cdot \sigma_Q = 104.6$

(b). Want $n \ni P(\bar{x} < \# \mid x_i \stackrel{iid}{\sim} N(800, 60^2)) = \frac{1}{370} = 0.0027$

and $P(\bar{x} < \# \mid x_i \stackrel{iid}{\sim} N(750, 60^2)) = \frac{1}{5} = 0.2$

$$\left.\begin{array}{l} \Rightarrow \Phi\left(\frac{\# - 800}{60/\sqrt{n}}\right) = 0.0027 = \Phi(-2.78) \\ \text{and } \Phi\left(\frac{\# - 750}{60/\sqrt{n}}\right) = 0.2 = \Phi(-0.84) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \# - 800 + 2.78 \times \frac{60}{\sqrt{n}} = 0 \\ \# - 750 + 0.84 \times \frac{60}{\sqrt{n}} = 0 \end{array}\right\} (*)$$

$(*) \Rightarrow n = 5.42 \approx 6$, $\# = 728.35 \simeq 728$ $\Rightarrow n = 6$ and LCL $\simeq 728$.

$\boxed{2.12}$

(a). $k_1 = \mu_Q + \delta/2 = 45 + 0.7/2 = 45.35$

$k_2 = \quad - \quad = 44.65$

$\mathcal{K} = (k_1 - \mu_0)/\sigma_Q = 0.35/0.7 = 0.5$

From table 4.5 of text, we get $\mathcal{H} = 5.07$

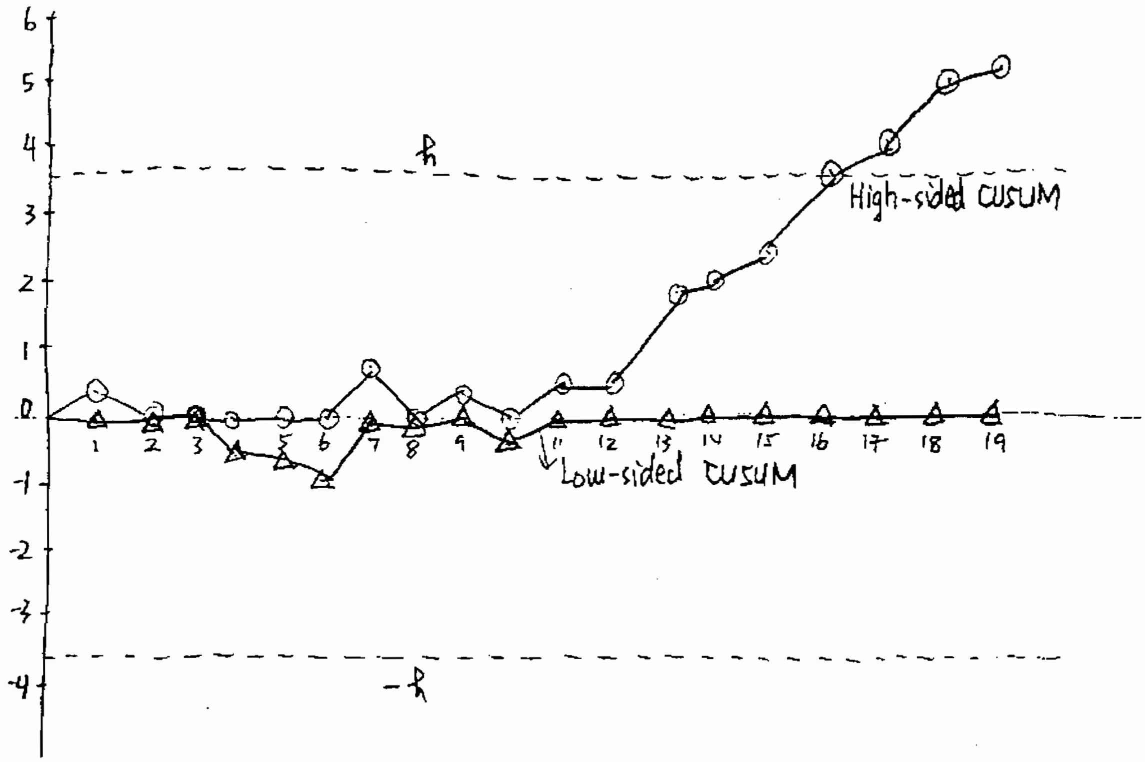
$\therefore h = \mathcal{H} \cdot \sigma_Q = (5.07)(.7) = 3.549$

$\therefore$ The required CUSUM scheme is given by the parameters:

$U_0 = 0$, $k_1 = 45.35$, $k_2 = 44.65$, $h = 3.549$

(b). Use a 0-headstart 2-sided CUSUM

$U_i = \max(0, (Q_i - k_1) + U_{i-1})$, $L_i = \min(0, (Q_i - k_2) + L_{i-1})$



Yes, the high-side CUSUM plot shows that there is a signal at the 16th sample. Low-side CUSUM plot doesn't reveal any signal

$\boxed{2.14}$ Running Gan's cusum ARL program.

$\underline{\mu_Q = 0}$:

$L_1(u) = 63.9$, $L_2(w) = 63.9$, $L_1(0) = 68.19$, $L_2(0) = 68.19$

$$ARL_{combined} = \frac{L_1(0)L_2(w) + L_2(0)L_1(u) - L_1(0)L_2(0)}{L_1(0) + L_2(0)} = 29.805$$

$\underline{\mu_Q = 1}$:

$L_1(u) = 4.07$, $L_2(w) = 10839.89$, $L_1(0) = 5.42$, $L_2(0) = 10861.33$

$$ARL_{combined} = 4.05728$$

$\boxed{2.25}$ Here, $X \sim \text{Poisson}(\lambda)$.

$$f_\lambda(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

To get ARL, use Markov Chain approach. Consider the 6 states:

	X_{t-4}	X_{t-3}	X_{t-2}	X_{t-1}
$S_1 =$	0	0	0	0
$S_2 =$	0	0	0	1
$S_3 =$	0	0	1	0
$S_4 =$	0	1	0	0
$S_5 =$	1	0	0	0

S_6 = alarm

The transition matrix is:

$$\begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \end{array} \\ \begin{bmatrix} e^{-\lambda} & \lambda e^{-\lambda} & 0 & 0 & 0 & 1-(e^{-\lambda}+\lambda e^{-\lambda}) \\ 0 & 0 & e^{-\lambda} & 0 & 0 & 1-e^{-\lambda} \\ 0 & 0 & 0 & e^{-\lambda} & 0 & 1-e^{-\lambda} \\ 0 & 0 & 0 & 0 & e^{-\lambda} & 1-e^{-\lambda} \\ e^{-\lambda} & 0 & 0 & 0 & 0 & 1-e^{-\lambda} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$ARL = L_1$

Let $\underset{\sim}{L} = (L_1, \ldots, L_5)^T$, L_i = mean # transitions required to move from state S_i to alarm (S_6).
R = upper left 5×5 submatrix of transition matrix. Compute $\underline{L} = (I_5 - R)^{-1} \cdot \underline{1}_5$

2.26

(a). $$\text{ARL} = \frac{1}{1 - p(-2) - p(-1) - p(0) - p(1) - p(2)}$$

(b). Define the states S_i's of a Markov Chain as follows :

$$S_i = \text{"}X_t = i\text{"}, \quad i = -2, -1, 0, 1, 2$$

and $S_a = \text{"}|X_t| \geq 3\text{"}$

The transition matrix is :

	S_{-2}	S_{-1}	S_0	S_1	S_2	S_a
S_{-2}	$p(0)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$	$1-\Sigma_0^4 p(j)$
S_{-1}	$p(-1)$	$p(0)$	$p(1)$	$p(2)$	$p(3)$	$1-\Sigma_{-1}^3 p(j)$
S_0	$p(-2)$	$p(-1)$	$p(0)$	$p(1)$	$p(2)$	$1-\Sigma_{-2}^2 p(j)$
S_1	$p(-3)$	$p(-2)$	$p(-1)$	$p(0)$	$p(1)$	$1-\Sigma_{-3}^1 p(j)$
S_2	$p(-4)$	$p(-3)$	$p(-2)$	$p(-1)$	$p(0)$	$1-\Sigma_{-4}^0 p(j)$
S_a	0	0	0	0	0	1

where R = upper left 5x5 submatrix of the above transition matrix

Now, the ARL will be the 3rd element of

$$\underset{\sim}{L} = (I^{5\times5} - R)^{-1}\, \underset{\sim}{1}^{5\times1} \qquad (\text{Since } \underset{\sim}{L} = L(-2), L(-1), L(0), L(1), L(2))$$

2.27

(a). $$L(u) = 1 \cdot \{P(X_2 \geq UCL) + P(X_2 \leq LCL)\} + \int_{LCL}^{UCL} (1 + L(v))\, f_{X_2|X_1}(v)\, dv$$

$\left(\text{where } f_{X_2|X_1}(v) = \text{pdf of a normal dsn with mean } \rho u, \text{ variance } \sigma^2\right)$

$$= 1 + \int_{LCL}^{UCL} L(v)\, f_{X_2|X_1}(v)\, dv$$

(b). $$\text{ARL} = 1 \cdot \{P(X_1 \leq LCL) + P(X_1 \geq UCL)\} + \int_{LCL}^{UCL} (1 + L(v))\, f_{X_1}(v)\, dv$$

$$= 1 + \int_{LCL}^{UCL} L(v) \cdot f_{X_1}(v)\, dv \quad \left[\text{where } f_{X_1}(v) = \text{pdf of } N\left(0, \frac{\sigma^2}{1-\rho^2}\right)\right]$$

2.28 Let $p_j = \dfrac{e^{-\lambda}\cdot\lambda^j}{j!}$, $j = 0, 1, 2, \ldots$

(a) Possible values of the cusum are $0, .5, 1, 1.5, \ldots, 3.5$ and 4.0 (or the alarm state.)
An appropriate transition matrix is

$$
P = \begin{array}{c} \\ 0 \\ 0.5 \\ 1.0 \\ 1.5 \\ 2.0 \\ 2.5 \\ 3.0 \\ 3.5 \\ \text{alarm} \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & .5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 & 3.5 & \text{Alarm} \end{array} \\
\begin{pmatrix}
p_0 & p_1 & 0 & p_2 & 0 & p_3 & 0 & p_4 & \sum_{j=5}^{\infty} p_j \\
p_0 & 0 & p_1 & 0 & p_2 & 0 & p_3 & 0 & \sum_{j=4}^{\infty} p_j \\
0 & p_0 & 0 & p_1 & 0 & p_2 & 0 & p_3 & \sum_{j=4}^{\infty} p_j \\
0 & 0 & p_0 & 0 & p_1 & 0 & p_2 & 0 & \sum_{j=3}^{\infty} p_j \\
0 & 0 & 0 & p_0 & 0 & p_1 & 0 & p_2 & \sum_{j=3}^{\infty} p_j \\
0 & 0 & 0 & 0 & p_0 & 0 & p_1 & 0 & \sum_{j=2}^{\infty} p_j \\
0 & 0 & 0 & 0 & 0 & p_0 & 0 & p_1 & \sum_{j=2}^{\infty} p_j \\
0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & \sum_{j=1}^{\infty} p_j \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

(b) A slightly modified version of the previous matrix is required, that is,

$$
\begin{array}{c} \\ 0 \\ .5 \\ 1.0 \\ 1.5 \\ 2.0 \\ 2.5 \\ 3.0 \\ 3.5 \\ \text{Alarm} \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & .5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 & 3.5 & \text{Alarm} \end{array} \\
\begin{pmatrix}
p_0 & p_1 & 0 & p_2 & 0 & 0 & 0 & 0 & 1-(p_0+p_1+p_2) \\
p_0 & 0 & p_1 & 0 & p_2 & 0 & 0 & 0 & 1-(p_0+p_1+p_2) \\
0 & p_0 & 0 & p_1 & 0 & p_2 & 0 & 0 & 1-(p_0+p_1+p_2) \\
0 & 0 & p_0 & 0 & p_1 & 0 & p_2 & 0 & 1-(p_0+p_1+p_2) \\
0 & 0 & 0 & p_0 & 0 & p_1 & 0 & p_2 & 1-(p_0+p_1+p_2) \\
0 & 0 & 0 & 0 & p_0 & 0 & p_1 & 0 & 1-(p_0+p_1) \\
0 & 0 & 0 & 0 & 0 & p_0 & 0 & p_1 & 1-(p_0+p_1) \\
0 & 0 & 0 & 0 & 0 & 0 & p_0 & 0 & 1-p_0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$\boxed{2.30}$

(a). $L(u) = 1 + L(0)\,F(k_1-u) + \int_0^h L(y)\, f(y+k_1-u)\,dy$

$$= 1 + L(0)\,(1-e^{u-k_1})\,I(k_1 \geq u) + \int_0^h L(y)\, e^{-(y+k_1-u)} \cdot I(y \geq u-k_1)\,dy$$

$$L'(u) = -L(0)\cdot e^{u-k_1}\cdot I(k_1 \geq u) + \int_0^h L(y)\, e^{-(y+k_1-u)} \cdot I(y \geq u-k_1)\,dy$$

Now, if $k_1 \geq u \geq 0$,

$$L'(u) = -L(0)\, e^{u-k_1} + \int_0^h L(y)\, e^{-(y+k_1-u)}\,dy$$

$$= L(u) - L(0) - 1 \quad \left(\because \text{In this case, } L(u) = 1 + L(0)(1-e^{u-k_1}) + \int_0^h L(y)\, e^{-(y+k_1-u)}\,dy \right) \quad \cdots (*)$$

If $k_1 < u$,

$$L(u) = 1 + \int_{u-k_1}^h L(y)\, e^{-(y+k_1-u)}\,dy$$

$$L'(u) = \int_{u-k_1}^h L(y)\, e^{-y-k_1+u}\,dy - L(u-k_1)$$

$$= L(u) - 1 - L(u-k_1) \quad \cdots (**)$$

$(*)$ & $(**)$ gives the desired result.

(b). Here, $k_1 = 1.5$, $h_1 = 4.0$, $m = 8$, $a_i = \frac{(2i-1)h}{2m}$, $w_i = \frac{h_1}{m} = 0.5$,

$f(x) = e^{-x}$, $F(x) = 1 - e^{-x}$ $\forall x > 0$.

Note $F(x) = 0$ if $x \leq 0$

$f(x) = 0$ if $x \leq 0$.

Using the argument in Section 2.3 of the notes,

$$R = \begin{pmatrix} F(k_1-a_1) + w_1 f(k_1) & w_2 f(a_2+k_1-a_1) & \cdots & w_m f(a_m+k_1-a_1) \\ F(k_1-a_2) + w_1 f(a_1+k_1-a_2) & w_2 f(k_1) & \cdots & w_m f(a_m+k_1-a_2) \\ \vdots & \vdots & & \vdots \\ F(k_1-a_m) + w_1 f(a_1+k_1-a_m) & w_2 f(a_2+k_1-a_m) & \cdots & w_m \cdot f(k_1) \end{pmatrix}$$

cont'd

$\boxed{2.30}$

(b) cont'd :

$$R = \begin{pmatrix}
(1-e^{-1.25})+.5e^{-1.5} & .5e^{-2} & .5e^{-2.5} & .5e^{-3} & .5e^{-3.5} & .5e^{-4} & .5e^{-4.5} & .5e^{-5} \\
(1-e^{-.75})+.5e^{-1} & .5e^{-1.5} & .5e^{-2} & .5e^{-2.5} & .5e^{-3} & .5e^{-3.5} & .5e^{-4} & .5e^{-4.5} \\
(1-e^{-.25})+.5e^{-.5} & .5e^{-1} & .5e^{-1.5} & .5e^{-2} & .5e^{-2.5} & .5e^{-3} & .5e^{-3.5} & .5e^{-4} \\
0 & .5e^{-.5} & .5e^{-1} & .5e^{-1.5} & .5e^{-2} & .5e^{-2.5} & .5e^{-3} & .5e^{-3.5} \\
0 & 0 & .5e^{-.5} & .5e^{-1} & .5e^{-1.5} & .5e^{-2} & .5e^{-2.5} & .5e^{-3} \\
0 & 0 & 0 & .5e^{-.5} & .5e^{-1} & .5e^{-1.5} & .5e^{-2} & .5e^{-2.5} \\
0 & 0 & 0 & 0 & .5e^{-.5} & .5e^{-1} & .5e^{-1.5} & .5e^{-2} \\
0 & 0 & 0 & 0 & 0 & .5e^{-.5} & .5e^{-1} & .5e^{-1.5}
\end{pmatrix}$$

(c). $m=8$, $k_1=1.5$, $h_1=4.0$

$$f^*(x) = \text{density of } Q_1-k_1 = Q_1-1.5$$

$$= \begin{cases} e^{-(1.5+x)} & , x \geq -1.5 \\ 0 & , \text{ow.} \end{cases}$$

replace continuous dsn $f^*(x)$ by discrete dsn concentrating on multiples of $h_1/m = 4/8 = .5$ from $-h_1=-4$ to $h_1=4$

$$p_{-8} = p_{-7} = p_{-6} = p_{-5} = p_{-4} = 0$$

$$p_{-3} = \int_{-1.5}^{-1.25} e^{-(1.5+x)}\,dx = -e^{-(1.5+x)}\Big|_{-1.5}^{2.5} = 1-e^{-.25} = .2212$$

$$p_{-2} = e^{-.25}-e^{-.75} = .3064$$

$$p_{-1} = e^{-.75}-e^{-1.25} = .1859$$

$$p_0 = e^{-1.25}-e^{-1.75} = .1127$$

$$p_1 = e^{-1.75}-e^{-2.25} = .0084$$

$$p_2 = e^{-2.25}-e^{-2.75} = .0415$$

$$p_3 = e^{-2.75}-e^{-3.25} = .0252$$

$$p_4 = e^{-3.25}-e^{-3.75} = .0153$$

$$p_5 = e^{-3.75}-e^{-4.25} = .0093$$

$$p_6 = e^{-4.25}-e^{-4.75} = .0056$$

$$p_7 = e^{-4.75}-e^{-5.25} = .0034$$

$$p_8 = \int_{3.75}^{\infty} e^{-(1.5+x)}\,dx = -e^{-(1.5+x)}\Big|_{3.75}^{\infty} = .0052$$

2.30

(c). cont'd

States : "no alarm to date & current high side CUSUM is $i(.5)$, $i = 0, \ldots, 7$"

and "alarm"

$$R = \begin{pmatrix} .8262 & .0684 & .0415 & .0252 & .0153 & .0093 & .0056 & .0034 \\ .7135 & .1127 & .0684 & .0415 & .0252 & .0153 & .0093 & .0056 \\ .5276 & .1859 & .1127 & .0684 & .0415 & .0252 & .0153 & .0093 \\ .2212 & .3064 & .1859 & .1127 & .0684 & .0415 & .0252 & .0153 \\ 0 & .2212 & .3064 & .1859 & .1127 & .0684 & .0415 & .0252 \\ 0 & 0 & .2212 & .3064 & .1859 & .1127 & .0684 & .0415 \\ 0 & 0 & 0 & .2212 & .3064 & .1859 & .1127 & .0684 \\ 0 & 0 & 0 & 0 & .2212 & .3064 & .1859 & .1127 \end{pmatrix}$$

This is quite similar to the R matrix in (b).

2.32

(a). $\mathrm{Var}\,X = E(X-EX)^2 = EX^2 - (EX)^2$

$E(X+1)^2 = EX^2 + 2EX + 1$

(b). Suppose $L_1 = E(RL)$

$L_2 = E(AP)$ where AP = # additional points for a signal if there is no alarm yet & current point is between 2 & 3-σ limits.

$M_1 = E(RL^2)$

$M_2 = E(AP^2)$

Then, from section 2.2 of the notes,

$$M_1 = 1^2 \cdot q_1 + E(1+AP)^2 q_2 + E(1+RL)^2 (1-q_1-q_2)$$

$$= q_1 + (M_2 + 2L_2 + 1) q_2 + (M_1 + 2L_1 + 1)(1-q_1-q_2)$$

$$M_2 = 1 \cdot (q_1+q_2) + E(1+RL)^2 (1-q_1-q_2)$$

$$= (q_1+q_2) + (M_1 + 2L_1 + 1)(1-q_1-q_2)$$

(c). $\mathrm{Var}\,(RL) = ERL^2 - (ERL)^2$

$$= M_1 - L_1^2$$