1. (Problem 1.11 of the notes)

   (a) Sheppard's corrected standard deviation is always smaller.

   (b) CONEST estimates of μ are close to and centered around the sample means. CONEST estimates of σ are consistently smaller than the sample standard deviation and the Sheppard's corrected standard deviation.

sample	xbar	conest.avg	range	s	s.conest	s.sheppardc
1	3.0	3.000	2	.707	.508	.563
2	2.4	2.260	1	.548		.396
3	.8	0.750	5	1.924	1.692	1.696
4	1.8	1.750	4	1.483	1.318	1.295
5	2.4	2.375	3	1.140	.944	.978
6	1.0	1.000	4	1.732	1.505	1.522
7	.4	0.375	2	.894	.758	.746
8	.8	0.750	3	1.304	1.131	1.130

   (c) The sample #2 has measurements (2, 2, 3, 3, 2) with $n = 5, R = 1, \overline{x} = 2.4$ and $s = .548$. Note that $x^* = 2, x^* + 1 = 3, n_{x^*} = 3, n_{x^*+1} = 2$, $\Delta_L \geq \Delta_U$ and $m = 3$ (see the notation on page 16 of the notes). From Table 1.3 of the notes,

   - $\alpha = .05 \quad \Rightarrow \quad (\Delta_L, \Delta_U) = (.748, .548)$
   - $\alpha = .10 \quad \Rightarrow \quad (\Delta_L, \Delta_U) = (.592, .339)$

   The interval for a normal mean based on integer-rounded data is given by $(x^* + .5 - \Delta_L, x^* + .5 + \Delta_U)$. Hence,

   - 95% CI for μ : (2.5 - .748, 2.5+.548) = (1.752, 3.048)
   - 90% CI for μ : (2.5 - .592, 2.5+.339) = (1.908, 2.839)

   Using Table 1.6 of the notes, when $n = 5, m = 3$, we have

   - $\alpha = .05$: $\Lambda_{1,3} = 1.516 \quad \Rightarrow \quad$ 95% CI for $\sigma = (0, 1.516)$
   - $\alpha = .10$: $\Lambda_{1,3} = 1.231 \quad \Rightarrow \quad$ 90% CI for $\sigma = (0, 1.231)$

   (d) The sample #1 has measurements (4, 3, 3, 2, 3) with $R = 2$. The MLE of (μ, σ) is given by $\widehat{\mu} = 3.0000$ and $\widehat{\sigma} = .5706$, with the maximized likelihood of -4.8007. CONEST yields approximate profile likelihood $\mathcal{L}^*(\mu)$ around the MLE as follows:

mu	L*(mu)
2.1	-7.5876
2.2	-7.1962
2.4	-6.4335
2.6	-5.6745
2.8	-5.0454
3.0	-4.8007
3.2	-5.0454
3.4	-5.6745
3.6	-6.4335
3.8	-7.1962
3.9	-7.5876
4.0	-7.9357

sigma	L**(sigma)
.2	-12.9473
.25	-7.7061
.275	-6.9471
.3	-6.3833
.4	-5.2090
.5	-4.8443
.5625	-4.7999
.6	-4.8139
.8	-5.1690
1.0	-5.7196
1.5	-7.1422
1.6	-7.4037
1.7	-7.6557

Note that when $n = 5, \alpha = .05$, $c(n, \alpha) = n \log\{t^2_{\alpha/2,(n-1)}/(n-1) + 1\} = 5.370167$ (see page 16 of the notes). Use this to obtain an approximate 95% CI for μ by $\{\mu|\mathcal{L}^*(\mu) > \sup \mathcal{L}(\mu, \sigma) - .5c(n, \alpha) = -7.485783\} \approx (2.2, 3.8)$. Similarly, use $d(n, \alpha) = 5.58$ from Table 1.4 of the notes to obtain an approximate 95% CI for σ by $\{\sigma|\mathcal{L}^{**}(\sigma) > \sup \mathcal{L}(\mu, \sigma) - d(n, \alpha) = -7.5899\} \approx (.275, 1.6)$.

(e) If the sample #9 has measurements (2, 2, 2, 2, 2), then $R = 0$ and $x^* = 2$.

- 95% CI for μ : $(x^* - \Delta, x^* + \Delta) = (1.5, 2.5)$
- 95% CI for σ : $(0, \Lambda_0) = (0, .666)$

2. (Problem 2.2 of the notes)

(a) We suppose $\mu = 21.0$oz and $\sigma = 1.0$oz.

- R chart with $\sigma = 1$ has limits:

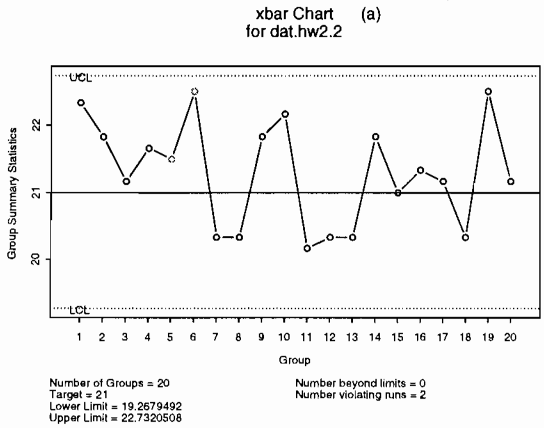
$$\begin{aligned}(UCL, CL, LCL) &= (D_2\sigma, d_2\sigma, D_1\sigma) \\ &= (4.36, 1.69, 0)\end{aligned}$$

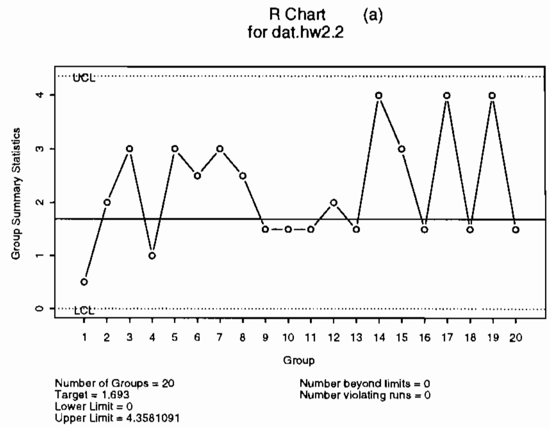
- $\overline{x}$ chart with $\sigma = 1, \mu = 21.0$ has limits:

$$\begin{aligned}(UCL, CL, LCL) &= (\mu + 3\sigma/\sqrt{n}, \mu, \mu - 3\sigma/\sqrt{n}) \\ &= (22.7, 21.0, 19.3)\end{aligned}$$

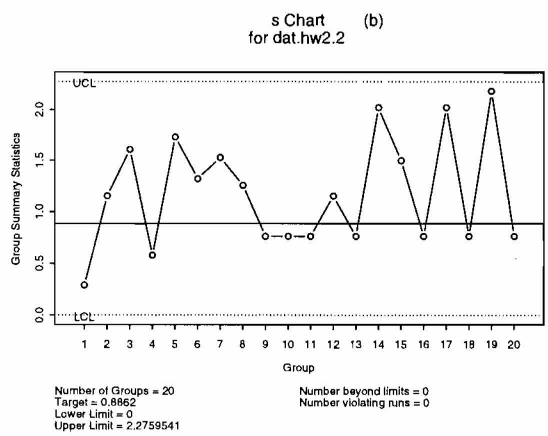
(b) s chart with $\sigma = 1$ has parameters:

$$\begin{aligned}(UCL, CL, LCL) &= (B_6\sigma, d_2\sigma, B_5\sigma) \\ &= (2.28, .89, 0)\end{aligned}$$





The 3-sigma alarm rule does not produce an "out of control" signal. The s and R charts are almost identical in pattern. (See the plots.)



(c)
- $\widehat{\sigma}$ based on $\overline{R}$: $\quad \widehat{\sigma}_1 = \overline{R}/d_2(3) = 2.30/1.693 = 1.359$
- $\widehat{\sigma}$ based on $\overline{s}$: $\quad \widehat{\sigma}_2 = \overline{s}/c_4(3) = 1.209/0.8862 = 1.365$

(d)
- Retrospective R chart with $\widehat{\sigma}_1$ has limits

$$\begin{aligned}(UCL, CL, LCL) &= (D_4\overline{R}, \overline{R}, D_3\overline{R}) \\ &= (5.92, 2.30, 0)\end{aligned}$$

- Retrospective $\overline{x}$ chart with $\widehat{\sigma}_1$ has limits:

$$\begin{aligned}(UCL, CL, LCL) &= (\overline{\overline{x}} + A_2\overline{R}, \overline{\overline{x}}, \overline{\overline{x}} - A_2\overline{R}) \\ &= (23.6, 21.3, 18.9)\end{aligned}$$

The process still appears "in control." (See the plots.)

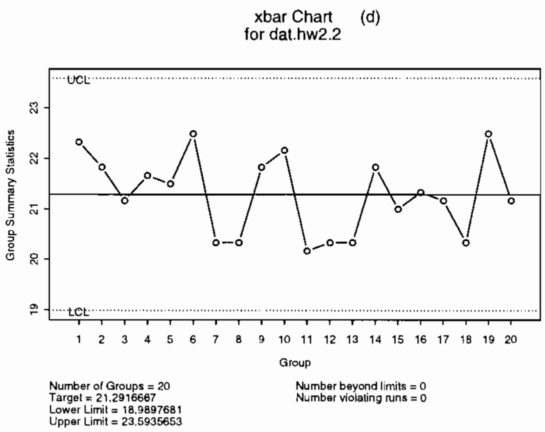
(e) • Retrospective s chart with $\widehat{\sigma}_2$ has limits

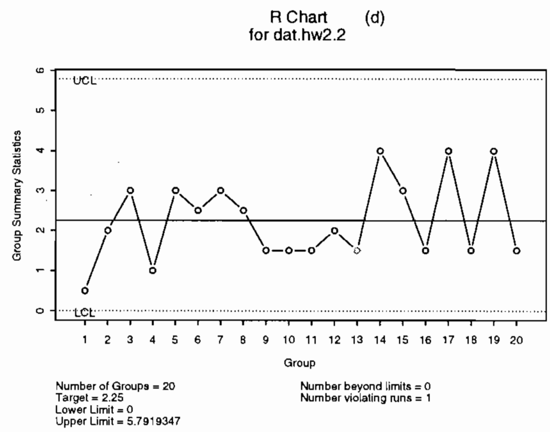
$$(UCL, CL, LCL) = (B_4\overline{s}, \overline{s}, B_3\overline{s}) = (3.11, 1.21, 0)$$

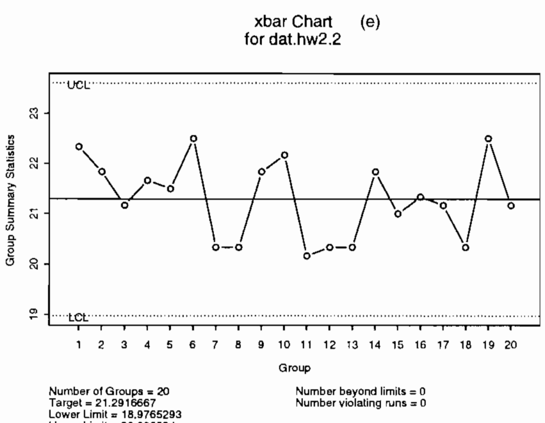
• Retrospective $\overline{x}$ chart with $\widehat{\sigma}_2$ has limits:

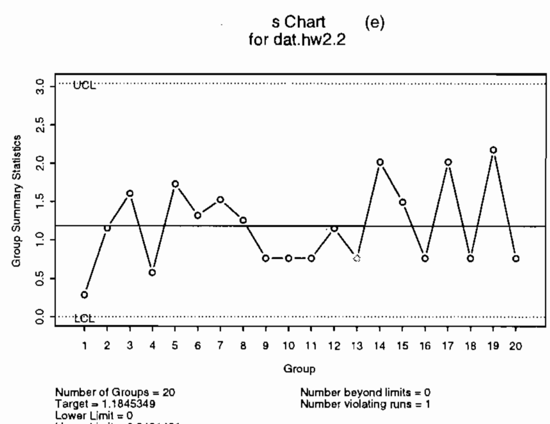
$$(UCL, CL, LCL) = (\overline{\overline{x}} + A_3\overline{s}, \overline{\overline{x}}, \overline{\overline{x}} - A_3\overline{s}) = (23.6, 21.3, 18.9)$$

Again, the process appears in control.









3. (Problem 2.19 of the notes)
Calculating s from the data you have simply inflated the standard deviation of the process if the process is unstable. Essentially 99.76% of your observations from a normal (μ, σ^2) are within $\mu \pm 3\sigma$. Since $\overline{x}$ and s are moderately good estimates of μ and σ, $\overline{x} \pm 3s$ would contain almost all the observations. In a sense you are computing $\sum(y_{ij} - y_{..})^2 = SST$ instead of $\sum(y_{ij} - y_{i.})^2 = SSE$. When you form your estimate of σ, this estimate will be much larger if the process is unstable. You need to estimate it not by $\sqrt{MST}$ but by $\sqrt{MSE}$.

4. (Problem 2.7 of the notes)

$$\widehat{p}_{pooled} = \frac{\sum x_i}{\sum n_i} = \frac{91}{225} = .4044$$

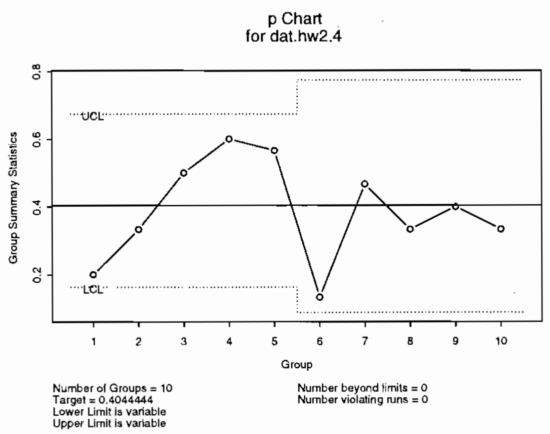
(a) Control limites for a p-chart for $n = 30$ are

$$(UCL_p, CL_p, LCL_p) = \left(\widehat{p} + 3\sqrt{\frac{\widehat{p}(1-\widehat{p})}{30}}, \widehat{p}, \widehat{p} - 3\sqrt{\frac{\widehat{p}(1-\widehat{p})}{30}}\right) = (.673, .404, .136).$$

For $n = 15$ these are

$$(UCL_p, CL_p, LCL_p) = (.784, .404, .024).$$

There is no apparent "out-of-control" pattern. Perhaps, the variation could be unstable over time, but there is not much data to know.



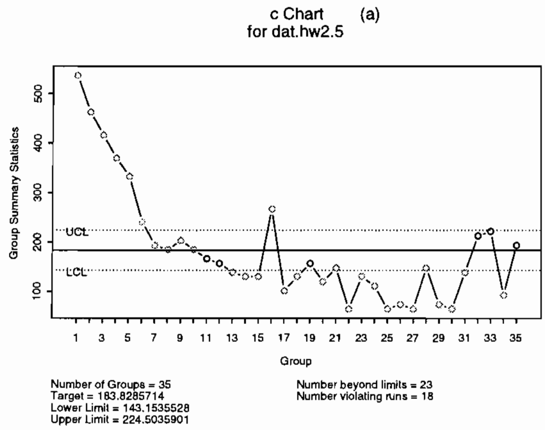
(b) This depends on what the company hopes to achieve and the history of the process. If a prior detection method gave a detection rate less than .20, this would be a decent improvement. Also, any given defective can be detected with a probability of $1 - (.4)^n$, which can be made arbitrarily close to 1 by redundancy.

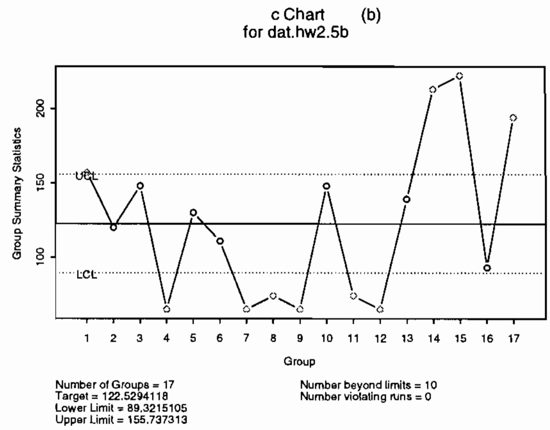
5. (Problem 2.6 of the notes)

(a) It seems as though the process is improving but it is not stable.

(b) No. (See the plot.)

(c) It is not enough to make either appear in control. Presumably we have an out of control process added to some random distribution on the values $\{-15, \cdots, 15\}$. Hence, the correct values should have less volatility. (Assuming a standard error of about 6 ($R/d_2(n) \approx 30/5 = 6$), it would have a new standard error of $\sqrt{\text{old var} - 36}$.)





6. (Problem 2.1 of the notes)

(a)

$$\widehat{\sigma}_A = \frac{\overline{R}_A}{d_2(10)} = \frac{2.0}{3.078} = .6497$$

$$\widehat{\sigma}_B = \frac{\overline{R}_B}{d_2(10)} = \frac{5.0}{3.078} = 1.6244$$

(b) The above estimates capture only $\sigma_{meas.}$, not the combined σ_{part} and $\sigma_{meas.}$.

(c) $\widehat{\sigma}_y = \frac{\overline{R}}{d_2(10)} = \frac{8}{3.078} = 2.599$.

- For $\overline{x}$, $200 \pm 3 \times 2.599/\sqrt{10} = \underline{(197.5, 202.5)}$
- For $\overline{R}$, $(D_3(10)R, D_4(10)R) = (.233 \times 8, 1.777 \times 8) = \underline{(1.784, 14.21)}$

(d) $\widehat{\sigma}_{diam} = \sqrt{\widehat{\sigma}_y^2 - \widehat{\sigma}_{meas.}^2} = \sqrt{2.599^2 - .6497^2} = 2.516$

(e) Note that $\widehat{\sigma}_y = \sqrt{2.516^2 + 1.6244^2} = 2.995$. Then

$$\begin{aligned} UCL_{\overline{x}} &= 200 + 3(3.00)/\sqrt{5} = 204.0 \\ LCL_{\overline{x}} &= 200 - 3(3.00)/\sqrt{5} = 196.0 \\ UCL_R &= D_2(5)\widehat{\sigma}_y = 14.75 \\ LCL_R &= D_1(5)\widehat{\sigma}_y = 0 \end{aligned}$$

7. (Problem 2.4 of the notes)

(a) Standards given 3σ control limits for a p chart with standard fraction nonconforming p and sample size n are $p \pm 3\sqrt{p(1-p)/n}$. These limits are computed as $(-.0029, .0031)$, or $(.0, .0031)$ when $p = 10^{-4}$ and $n = 100$. Let

$X \sim \text{Binomial}(n = 100, p = 10^{-4})$ and note that

$$\Pr(\widehat{p} = x/100) = \Pr(X = x) = \binom{100}{x} (.0001)^x (.9999)^{100-x}.$$

Since a single nonconforming unit will trigger an alarm,

$$\Pr(\text{alarm}) = \Pr(\widehat{p} > .0031) = \Pr(X > .31) = 1 - \Pr(X = 0) = 1 - (.9999)^{100} = .00995$$

and thus, all-OK ARL is

$$\begin{aligned} \text{ARL} &= EY \qquad\qquad Y \sim \text{Geometric}(.00995) \\ &= (.00995)^{-1} \\ &= 100.5. \end{aligned}$$

(b) If $p = .0002$ then

$$\Pr(\widehat{p} = x/100) = \Pr(X = x) = \binom{100}{x} (.0002)^x (.9998)^{100-x}.$$

Since, again, a single nonconforming unit will trigger an alarm, $\Pr(\text{alarm}) = 1 - (.9998)^{100} = .0198$.
Hence, ARL is $1/.0198 = 50.5$.

(c) For small p, we are likely to have an alarm with only a single defect, which would defeat the purpose of control charting unless we have large samples.

8. (Problem 4.4 of V&J)

(a) Note that $\overline{MR} = 2.06$ and use $\widehat{\sigma} = \overline{MR}/d_2(2) = 1.826$.

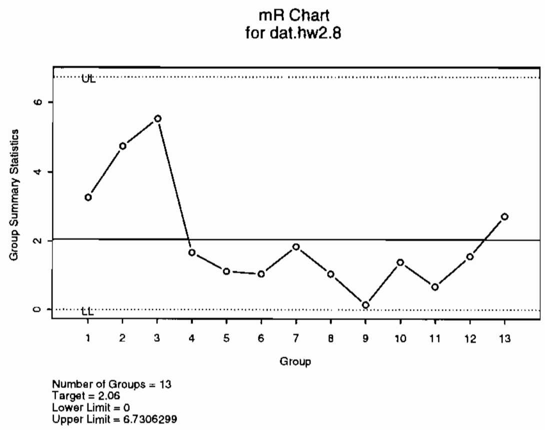
- For $\overline{x}$,

$$\begin{aligned} CL_{\overline{x}} &= \overline{x} = 3.38 \\ UCL_{\overline{x}} &= CL_x + 3\widehat{\sigma} = 8.86 \\ LCL_{\overline{x}} &= CL_x + 3\widehat{\sigma} = -2.10 \end{aligned}$$

- For MR,

$$\begin{aligned} UCL_{MR} &= D_2(2)\widehat{\sigma} = 6.73 \\ LCL_{MR} &= 0 \qquad\qquad \text{(off the chart)} \end{aligned}$$

(b) The range seems to stay below the center line for eight consecutive points. This may indicate an out-of-control process.



(c) The MR chart, since the x-chart is based on a stable σ.

(d) $\overline{x} = 3.384, \hat{\sigma} = \overline{MR}/d_2(2) = 1.826$.

9. (Problem 4.31 of V&J)

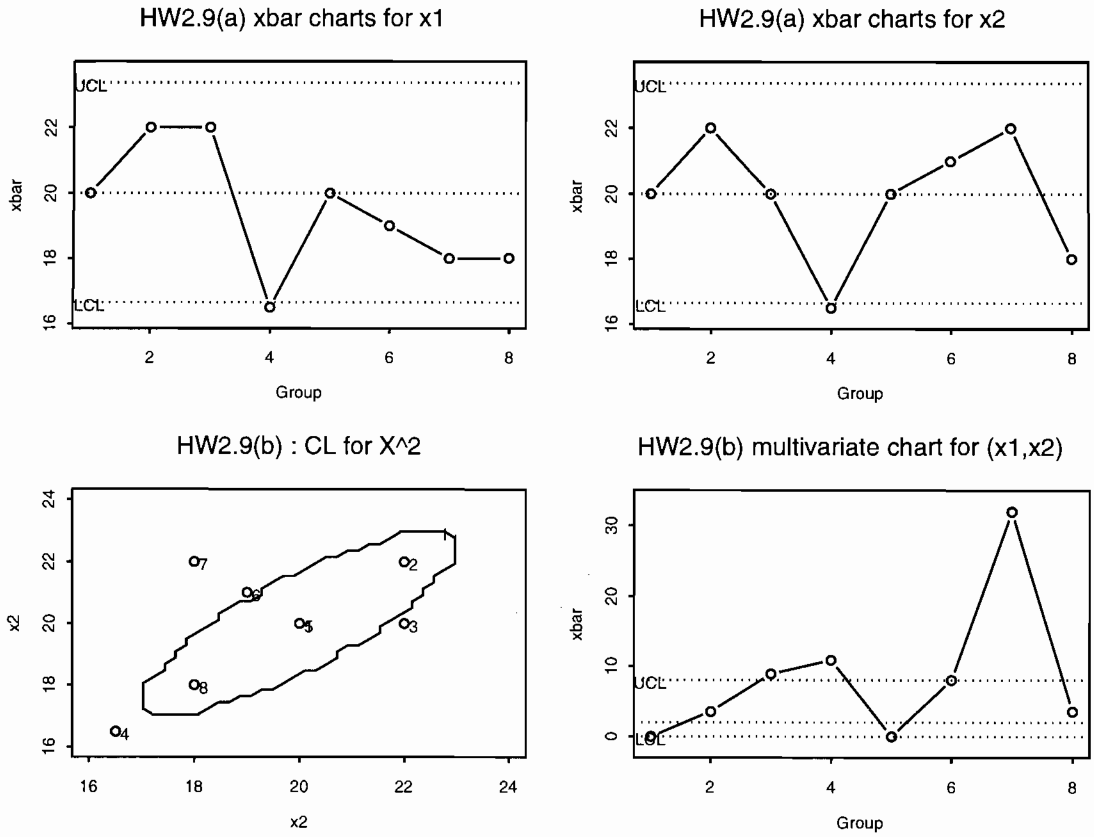
(a) $UCL_{\overline{x}} = 20 + 3\sqrt{5/4} = 23.4$, $LCL_{\overline{x}} = 20 - 3\sqrt{5/4} = 16.5$. Only subgroup 4 is outside the limits. (See the plot.)

(b)

$$X^2 = n\left[\left(\frac{\overline{x}_1 - \mu_1}{\sigma_1}\right)^2 + \left(\frac{\overline{x}_2 - \mu_2}{\sigma_2}\right)^2\right] \quad \sim \quad \chi^2_2$$

Note that $p = 2$, $CL_{X^2} = p$, and $UCL_{X^2} = p + 3\sqrt{2p} = 8$. Hence any point (x_1, x_2) out of the contour $X^2 = 8$ is considered as 'out-of-control'. See the plot and the following table.

x1	x2	X^2	out?
20.0	20.0	.000000	N
22.0	22.0	3.555556	N
22.0	20.0	8.888889	Y
16.5	16.5	10.888889	Y
20.0	20.0	.000000	N
19.0	21.0	8.000000	N
18.0	22.0	32.000000	Y
18.0	18.0	3.555556	N



(c) Points 3, 7 and perhaps 6 are outside the control limit but not detected in (a). This can happen because the correlation (being positive) creates a rather large area where points may lie in circumscribed rectangle yet not in the ellipse.

10. (Problem 4.36 of V&J)

(a) The sample means are (1.49975, .324125, .0794375) and the sample variance-covariance matrix is computed as follows.

	A	B	C
A	2.214286e-006	-3.571429e-008	4.107143e-007
B	-3.571429e-008	1.964286e-007	1.875000e-007
C	4.107143e-007	1.875000e-007	4.598214e-007

(b) The variance-covariance matrix calculated by the Holmes and Mergen method is computed as follows.

	A	B	C
A	1.285714e-006	7.142857e-008	4.285714e-007
B	7.142857e-008	2.142857e-007	2.321429e-007

```
C  4.285714e-007 2.321429e-007 4.821429e-007
```

(c) Values for plotting (in order) on a control chart are as follows. Note that $UCL_{X^2} = p + 3\sqrt{2p} = 10.3485$. The chart appears stable.

part	A	B	C	X^2
1	1.499	.3250	.0800	4.501426
2	1.498	.3235	.0780	4.532478
3	1.500	.3240	.0790	1.245406
4	1.502	.3240	.0800	4.261882
5	1.502	.3240	.0795	5.690272
6	1.499	.3245	.0795	1.338561
7	1.499	.3240	.0800	6.497624
8	1.499	.3240	.0795	1.644645