Second Order Linear Differential Equations

We've looked at special cases of the differential equation

$$ay'' + by' + cy = 0,$$

where a, b, and c are constants.

If we consider the more general homogeneous equations

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are functions of t, when can we be assured that a solution exists?

> **Theorem.** *Let I be an interval that contains the point t_0, and suppose the functions $p(t)$, $q(t)$ and $g(t)$ are continuous on I. Then there is exactly one solution to the initial value problem*
>
> $$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$
>
> *valid in the interval I.*

Example. *Find the largest interval in which the solution to the initial value problem*

$$(\cos t)y'' + ty' + 5y = 1, \qquad y(0) = 2. \qquad y'(0) = 3.$$

is certain to exist

Rewrite the differential equation in the form appearing in the theorem

$$y'' + \frac{t}{\cos t}y' + \frac{5}{\cos t}y = \frac{1}{\cos t}.$$

The coefficient functions

$$\frac{t}{\cos t}, \qquad \frac{5}{\cos t}, \qquad \frac{1}{\cos t}$$

are continuous in the interval $(-\pi/2, \pi/2)$. This interval contains $t = 0$.

Therefore, there is a unique solution to the IVP on the interval $(-\pi/2.\pi/2)$.

Anytime we have two solutions of a second order homogeneous linear differential equation, their linear combination is also a solution.

This result is sometimes called the **principle of superposition.**

Theorem. *If y_1 and y_2 are solutions to the differential equation*

$$y'' + p(t)y' + q(t)y = 0,$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any choice of constants c_1, c_2.

The superposition principle results from the basic differentiation rules

$$\frac{d}{dt}(y_1(t) + y_2(t)) = \frac{d}{dt}y_1(t) + \frac{d}{dt}y_2(t), \qquad \frac{d}{dt}(Cy(t)) = C\frac{d}{dt}y(t), \qquad C \in \mathbb{R}.$$

Key Questions:

If we have two solutions y_1 and y_2 to the equation

$$y'' + p(t)y' + q(t)y = 0,$$

1. Is there a choice of constants for which $y(t) = c_1y_1(t) + c_2y_2(t)$ also satisfies some arbitrary initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0' \qquad ?$$

2. Can we get all the solutions through some linear combination

$$c_1y_1 + c_2y_2, \qquad c_1, c_2 \in \mathbb{R} \qquad ?$$

To answer these questions, we need to define a few facts from linear algebra.

The **determinant** of the **matrix**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

is denoted / defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

How do determinants and matrices show up in differential equations?

In order for a linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

to satisfy the initial conditions, $y(t_0) = y_0,\ y'(t_0) = y_0'$, we need

$$\begin{aligned} y_0 &= y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ y_0' &= y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0). \end{aligned}$$

If we solve this system via substitution, after some elementary algebra, we find

$$c_1 = \frac{y_0 \cdot y_2'(t_0) - y_0' \cdot y_2(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}, \qquad c_2 = \frac{-y_0 \cdot y_1'(t_0) + y_0' \cdot y_1(t_0)}{y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0)}.$$

Using determinants, this can be written as

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}.$$

The denominator in each case is the same:

$$W(y_1, y_2)(t_0) := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0) y_2'(t_0) - y_1'(t_0) y_2(t_0).$$

We call $W(y_1, y_2)(t_0)$ the **Wronskian** of y_1 and y_2 evaluated at t_0.

Our calculations above show that, so long as the denominators in the expressions for c_1, c_2 above are nonzero, we can always satisfy the initial conditions.

Theorem. *Suppose that y_1 and y_2 are two solutions to*

$$y'' + p(t)y' + q(t)y = 0,$$

and that the Wronskian

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

is not zero at the point t_0 where the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

are assigned. Then there is a choice of the constants c_1, c_2 for which

$$y = c_1 y_1(t) + c_2 y_2(t)$$

satisfies the differential equation and the initial condiditions.

This means that we can always find a solution to an IVP

$$y'' + p(t)y' + q(t)y = 0, \qquad\qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

with a linear combination of two particular solutions, $y_1(t)$ and $y_2(t)$, to the differential equation, provided $W(y_1, y_2)(t_0) \neq 0$.

In fact, as shown on the last page, the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t),$$

where c_1 and c_2 are defined by

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}.$$

Theorem. *If y_1 and y_2 are two solutions of the differential equation*

$$y'' + p(t)y' + q(t)y = 0, \tag{1}$$

and if there is a point t_0 where the Wronskian of y_1 and y_2 is nonzero, then the family of solutions

$$y = c_1 y_1(t) + c_2 y_2(t)$$

includes each solution to (1) for some choice of constants c_1, c_2.

Proof. Suppose you have an arbitrary function $\theta(t)$ satisfying the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

we need to see that

$$\theta(t) = k_1 y_1(t) + k_2 y_2(t), \qquad \text{for some constants } k_1, k_2.$$

If y_1 and y_2 are two particular solutions to (1) and t_0 is a point where the Wronskian of y_1 and y_2 is nonzero, the theorem on the last page implies that we can choose c_1 and c_1 so that a solution to the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = \theta(t_0), \qquad y'(t_0) = \theta'(t_0) \tag{2}$$

is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Therefore, the two functions

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \qquad \text{and} \qquad \theta(t)$$

are both solutions to the IVP (2). By the uniqueness of solutions to the IVP (2) from the first theorem in today's notes, θ and y are really the same function.

That is,

$$\theta(t) = y(t) = c_1 y_1(t) + c_2 y_2(t).$$

□

Example. *Find the Wronskian of the functions*

$$e^{-2t}, \qquad te^{-2t}$$

at $t = 0$.

We form matrix with *the two functions in the first row* and their *respective derivatives in the second row*:

$$\begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{pmatrix}.$$

To compute the Wronskian at $t = 0$, we evaluate at the determinant at $t = 0$:

$$W(e^{-2t}, te^{-2t})(0) = \begin{vmatrix} e^{-2\cdot 0} & 0 \cdot e^{-2\cdot 0} \\ -2e^{-2\cdot 0} & e^{-2\cdot 0} - 2 \cdot 0 \cdot e^{-2\cdot 0} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1 \cdot 1 - (-2) \cdot 0 = 1.$$

Why is the Wronskian important? It tells us when two particular solutions form a fundamental set of solutions (i.e., it tells us when we can expect to get every solution through a linear combination of two solutions.)

We can also extend the notions (of the determinant / Wronskian) to higher order differential equations.

Example. *We proved yesterday that, if r_1 and r_2 are two different, real roots of the quadratic equation*

$$ar^2 + br + c = 0,$$

then, for any choice of constants c_1 and c_2,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is a solution to the differential equation

$$ay'' + by' + cy = 0.$$

Do the solutions $e^{r_1 t}$ and $e^{r_2 t}$ form a fundamental set of solutions for the differential equation on the interval $(-\infty, \infty)$?

That is, are there any solutions that are not of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

for some choice of constants c_1 and c_2?

The Wronskian of $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ equals

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \\ &= r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} \\ &= (r_1 - r_2) e^{(r_1+r_2)t}. \end{aligned}$$

Since r_1 and r_2 are different, $r_1 \neq r_2$, so that $r_1 - r_2 \neq 0$.

No matter which t we choose in the interval $(-\infty, \infty)$,

$$W(e^{r_1 t}, e^{r_2 t}) \neq 0.$$

By the last theorem above, this means that $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ form a fundamental set of solutions.

In other words, every solution, $y(t)$, to the differential equation $ay'' + by' + cy = 0$ is of the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

for some choice of constants c_1 and c_2.