Exact Equations and Integrating Factors

We have the following methods available to use in solving differential equations:

- Inspection, or solving via a clever trick,
- Integration factors on first order linear equations,
- Separation of variables on separable equations.

We now introduce a new procedure, applicable on **exact equations**.

Suppose we have an ODE of the form

$$\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0.}$$

Suppose, in addition, we can find a function $\psi(x,y)$ such that

$$\boxed{\frac{\partial \psi}{\partial x} = M(x,y) \qquad \text{and} \qquad \frac{\partial \psi}{\partial y} = N(x,y).}$$

Then, by the chain rule (a version for multivariable functions from calc. III),

$$\begin{aligned} 0 = M(x,y) + N(x,y)\frac{dy}{dx} &= \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} \\ &= \frac{d}{dx}\psi[x, y(x)], \end{aligned}$$

where we used the chain rule for the final equality.

The extreme right and left sides of the above equality imply

$$\boxed{\frac{d}{dx}\psi\Big(x, y(x)\Big) = 0.}$$

Integrate both sides with respect to x to find a solution given implicitly by

$$\boxed{\psi(x,y) = C.}$$

Definition. *The first order equation*

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be ***exact*** *(in some region of the xy-plane) if there is some function* $\psi(x,y)$ *with continuous partial derivatives such that*

$$\frac{\partial\psi(x,y)}{\partial x} = M(x,y), \qquad \textit{and} \qquad \frac{\partial\psi(x,y)}{\partial y} = N(x,y).$$

Example. *Solve the differential equation*

$$3x^2 + y^2 + 2xyy' = 0. \tag{1}$$

Define

$$\psi(x,y) = x^3 + xy^2.$$

Then

$$\frac{\partial\psi}{\partial x} = 3x^2 + y^2, \qquad \text{and} \qquad \frac{\partial\psi}{\partial y} = 2xy.$$

Equation (1) can be written as

$$(3x^2 + y^2) + (2xy)\frac{dy}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{2}$$

By the chain rule for multi-variable functions, (2) can be written as

$$\frac{d}{dx}(x^3 + xy^2) = 0 \qquad \text{or, equivalently,} \qquad \frac{d\psi}{dx} = 0. \tag{3}$$

Here, since we are assuming y is a function of x, the calculations above also follow by **implicit differentiation**.

To find the solution to (1), we integrate (3) with respect to x, to obtain

$$x^3 + xy^2 = C \qquad \text{or, equivalently,} \qquad \psi(x,y) = C.$$

This equation gives solutions $y = y(x)$ to (1) **implicitly**.

The challenge in solving exact equations lies in finding the function $\psi(x, y)$. We will make the method more precise in later examples. For now, it may look mysterious.

Exactness is equivalent to the existence of the function $\psi(x, y)$

Theorem. *Suppose* $M, N, \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ *are continuous in the rectangular region*

$$D := \{(x, y) \mid \alpha < x < \beta,\ \gamma < y < \delta\}.$$

Then there exists a function $\psi(x, y)$ *such that*

$$\frac{\partial \psi}{\partial x} = M(x, y), \qquad \frac{\partial \psi}{\partial y} = N(x, y) \tag{4}$$

if and only if
" $\Longleftrightarrow$ "

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y). \tag{5}$$

Sketch of Proof.

$\Longrightarrow$:
Suppose there exists a function $\psi(x, y)$ such that (4) is satisfied. Then, taking mixed partial derivatives of these equations, we obtain

$$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial M}{\partial y}, \qquad \text{and} \qquad \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial N}{\partial x}. \tag{6}$$

Recall that if all functions involved are continuous, then mixed partial derivatives are equal. In this case, from (6), we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}. \tag{7}$$

$\Longleftarrow$:
Suppose that (5) is satisfied. Then, for any $(x_0, y_0) \in D$, the function

$$\psi(x, y) = \int_{x_0}^{x} M(s, y_0)\ ds + \int_{y_0}^{y} N(x, t)\ dt$$

satisfies equation (4). $\square$

General Method for Solving Exact Equations

1. Identify M, N, by writing the equation in the form
$$M(x,y) + N(x,y)y' = 0,$$
and show exactness of the equation by verifying that
$$\frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y).$$

2. Integrate
$$\frac{\partial \psi}{\partial x} = M(x,y)$$
with respect to x, including an arbitrary function $h(y)$ instead of a constant.

3. Differentiate with respect to y the function $\psi(x,y)$ from step 2 and set
$$\frac{\partial \psi}{\partial y} = N(x,y).$$

4. Solve for $h(y)$ using the equation from step 3.

5. The general solution to the differential equation is given implicitly by
$$\psi(x,y) = C.$$

Throughout this process, we repeatedly use

$$\frac{\partial \psi}{\partial x} = M(x,y), \quad \frac{\partial \psi}{\partial y} = N(x,y) \qquad \text{and} \qquad \frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y).$$

It may be helpful to write these equations at the top of the page when solving an exact equation.

Remark: In step 2, it may be easier to integrate
$$\frac{\partial \psi}{\partial y} = N(x,y)$$
with respect to y and interchange x and y in successive steps.

Example. *Solve the differential equation*

$$y\cos x + 2xe^y + (\sin x + x^2e^y - 1)y' = 0. \tag{8}$$

This equation is of the form

$$M(x,y) + N(x,y)y' = 0,$$

with

$$M(x,y) = y\cos x + 2xe^y, \qquad N(x,y) = \sin x + x^2e^y - 1.$$

Therefore,

$$\frac{\partial M}{\partial y} = \cos x + 2xe^y, \qquad \frac{\partial N}{\partial x} = \cos x + 2xe^y,$$

so we conclude that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Therefore, equation (8) is exact. So there exists $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) = y\cos x + 2xe^y \tag{9}$$

$$\frac{\partial \psi}{\partial y} = N(x,y) = \sin x + x^2e^y - 1 \tag{10}$$

If we integrate both sides of (9) with respect to x, we find

$$\psi(x,y) = y\sin x + x^2e^y + h(y),$$

for some differentiable function $h(y)$.

Taking the partial derivative of $\psi(x,y)$ with respect to y, we find

$$\frac{\partial \psi}{\partial y} = \sin x + x^2e^y + h'(y).$$

By (10),

$$\frac{\partial \psi}{\partial y} = N(x, y) = \sin x + x^2 e^y - 1,$$

so that

$$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

Therefore,

$$h'(y) = -1,$$

and so

$$h(y) = -y.$$

So we have

$$\psi(x, y) = y \sin x + x^2 e^y - y.$$

By differentiating $\psi(x, y)$ implicitly with respect to x, it is easy to verify that the differential equation

$$y \cos x + 2xe^y + (\sin x + x^2 e^y - 1)y' = 0$$

is equivalent to

$$\frac{d}{dx} \psi(x, y) = 0.$$

Therefore, the general solution to the differential equation is given implicitly by

$$\psi(x, y) = C,$$

or, equivalently,

$$y \sin x + x^2 e^y - y = C.$$

Sometimes equations which are not exact can be made exact by multiplying by an integration factor.

Example. *Solve the differential equation*

$$(xy^2 + 4x^2y) + (3x^2y + 4x^3)y' = 0. \tag{11}$$

This equation is not exact since

$$\frac{\partial M}{\partial y} = 2xy + 4x^2, \qquad \frac{\partial N}{\partial x} = 6xy + 12x^2.$$

If we multiply both sides of (11) by $\mu(x, y) = \dfrac{y}{x}$, the equation becomes

$$(y^3 + 4xy^2) + (3xy^2 + 4x^2y)y' = 0.$$

This equation is exact since

$$\frac{\partial M}{\partial y} = 3y^2 + 8xy = \frac{\partial N}{\partial x}.$$

Proceeding as in the previous example, we find that

$$\psi(x, y) = xy^3 + 2x^2y^2 + h(y)$$

and so

$$3xy^2 + 4x^2y = N(x, y) = \frac{\partial \psi}{\partial y} = 3xy^2 + 4x^2y + h'(y).$$

Therefore,

$$h(y) = 0,$$

and so

$$\psi(x, y) = xy^3 + 2x^2y^2.$$

Therefore, solutions of the differential equation are given implicitly by

$$xy^3 + 2x^2y^2 = C.$$

Finding an integration factor that will transform a non-exact equation into an exact equation is difficult in general.

If we assume that the integration factor, μ, is a function of only one variable, say x, then we can try to find μ by writing down what it means for

$$\mu(x)M(x,y) + \mu(x)N(x,y)y' = 0$$

to be exact. Namely, that

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N),$$

or, equivalently, that

$$\mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x} + N\frac{d\mu}{dx}.$$

Solving for $\frac{d\mu}{dx}$, this implies

$$\frac{d\mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\mu. \tag{12}$$

Example. *Solve the differential equation*

$$3xy + y^2 + (x^2 + xy)y' = 0.$$

Condition (12) can be written as

$$\frac{d\mu}{dx} = \frac{\mu}{x} \qquad \Longrightarrow \qquad \frac{\mu'}{\mu} = \frac{1}{x} \qquad \Longrightarrow \qquad \frac{d}{dx}\ln\mu(x) = \frac{1}{x}, \qquad \mu(x) > 0.$$

Thus, an integration factor $\mu(x) = x$ can be chosen. After multiplying through by $\mu(x) = x$ in the differential equation, we obtain the exact equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

Proceeding as above we show that solutions are given implicitly by

$$x^3y + \frac{1}{2}x^2y^2 = C.$$