Existence and Uniqueness of Solutions to Linear and Nonlinear Differential Equations

The following theorem tells us when we should expect to find a unique solution to an IVP involving a first order linear equation.

Theorem (Existence & Uniqueness for First Order Linear Equations)**.** *The initial value problem*

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0 \tag{1}$$

has a unique solution in the interval $I = \{t \mid \alpha < t < \beta\} = (\alpha, \beta)$ *provided*

$$\boxed{\textit{The functions } p(t) \textit{ and } g(t) \textit{ are continuous on } I.}$$

$$\boxed{\textit{The interval } I \textit{ contains the point } t = t_0 \textit{ corresponding to the initial value.}}$$

Idea of Proof. If $p(t)$ is continuous, we can always define

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(s)\ ds\right)$$

and multiply both sides of (1) by $\mu(t)$. Recognizing (as before) that, for this choice of $\mu(t)$,

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t) \qquad \Longrightarrow \qquad \frac{d}{dt}(\mu(t)y) = \mu(t)g(t),$$

we integrate both sides with respect to t to obtain

$$\mu(t)y = \int \mu(t)g(t)\ dt + C.$$

We can then solve for $y(t)$ to obtain, in general,

$$y(t) = \frac{1}{\mu(t)} \int \mu(t)g(t)\ dt + \frac{C}{\mu(t)}.$$

What remains is to choose C so that the condition $y(t_0) = y_0$ is satisfied (and to define y in terms of a definite integral). It turns out that the function

$$y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s)g(s)\ ds + \frac{y_0}{\mu(t)}.$$

satisfies the initial condition. □

Example. *Find an interval in which the IVP*

$$ty' + 2y = 4t^2, \qquad y(1) = 2$$

has a unique solution.

We first recast the equation in the form given in the theorem:

$$y' + \frac{2}{t}y = 4t.$$

Here

$$p(t) = \frac{2}{t}, \qquad g(t) = 4t.$$

We need to find an interval in t

1. $\underline{\text{On which } p \text{ and } g \text{ are continuous,}}$
2. $\underline{\text{Containing } t = 1.}$

This is accomplished by choosing

$$\boxed{I = \{t \mid t > 0\} = (0, \infty)}$$

By the theorem, this IVP has a unique solution on I. In fact, we can solve this IVP using the method of integrating factors to find that

$$y(t) = t^2 + \frac{1}{t^2}, \qquad t > 0,$$

Example. *What does the theorem say about the initial value problem*

$$ty' + 2y = 4t^2, \qquad y(0) = 0?$$

$\underline{\text{Nothing.}}$

$\underline{\text{The theorem does not apply.}}$

Why?

When we rewrite the equation in standard form,

$$y' + \frac{2}{t}y = 4t,$$

the function $p(t) = \frac{2}{t}$ is discontinuous at $t = 0$.

There is no interval that can be chosen about $t = 0$ so that $p(t)$ is continuous.

$$\boxed{\text{The hypotheses of the theorem are not satisfied}}$$

This does not mean that a solution does not exist:

$$y = t^2$$

satisfies the differential equation and the initial condition.

The theorem we stated above applies only to linear differential equations that can be written in the form

$$y' + p(t)y = g(t).$$

What can we say about more general equations, namely nonlinear equations that can be written in the form

$$\frac{dy}{dt} = f(t, y),$$

for some function f.

Theorem (General Existence and Uniqueness Theorem for First Order IVPs)**.**

Let the functions $f(t, y)$ *and* $\dfrac{\partial f(t, y)}{\partial y}$ *be continuous in some rectangle*

$$\alpha < t < \beta, \qquad \gamma < y < \delta$$

containing the point (t_0, y_0)*. Then, there exists a nontrivial interval*

$$t_0 - h < t < t_0 + h,$$

contained in $\alpha < t < \beta$*, such that the initial value problem*

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{2}$$

has a unique solution within the interval $t_0 - h < t < t_0 + h$*.*

Proof. We defer the proof to a more advanced course in differential equations.

□

Example. *Is there an interval in which the IVP*

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \qquad y(0) = -1$$

has a unique solution?

First note that this ODE cannot be written in the form

$$y' + p(t)y = g(t).$$

That is, this ODE is not linear. So we *cannot* apply the existence and uniqueness theorem for first order linear equations.

We *can* apply the more general theorem above – it works for linear and nonlinear equations.

Both

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$$

are continuous everywhere except the line $y = 1$.

Luckily, our initial value $y(0) = -1$, corresponding to the ordered pair $(0, -1)$ is not on the line $y = 1$.

Thus, the theorem says that there is a rectangle about $(0, -1)$ in which

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \qquad y(0) = -1$$

has a unique solution.

In fact, this equation is separable, and we can show explicitly that the solution

$$y = 1 - \sqrt{x^3+2x^2+2x+4}, \qquad x > -2$$

is the unique solution in the rectangle satisfying the initial condition $y(0) = -1$ is

$$\{(x,y) \mid x > -2,\ y < 1\}.$$

An important corollary follows from these existence and uniqueness theorems.

Corollary. *Suppose* $y' = f(t, y)$*, and the functions* f *and* $\frac{\partial f}{\partial y}$ *are continuous in some rectangle* R*. Then two solutions with different initial conditions cannot intersect in* R*.*

Proof. Suppose y_1 and y_2 are two different solutions to the ODE

$$y' = f(t, y).$$

If they intersect at a point, say

$$y_1(t_0) = y_2(t_0),$$

then y_1 and y_2 are two *different* solutions to the same IVP, namely

$$y' = f(t, y), \qquad y_1(t_0) = y_2(t_0).$$

This contradicts the uniqueness part of the theorem on IVPs. $\square$

Example. *The IVP*

$$y' = y^{1/3}, \qquad y(0) = 0$$

Has many different solutions, including, for $t \geq 0$,

$$y_1 = \left(\frac{2}{3}t\right)^{3/2}, \qquad y_2 = -\left(\frac{2}{3}t\right)^{3/2}, \qquad y_3 = 0.$$

The nonlinear ODE appearing here does not satisfy the hypotheses of the general theorem for existence and uniqueness because

$$\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$$

is not continuous in any rectangle that includes the point $(x, y) = (0, 0)$ appearing in the initial condition.

We summarize some of the important differences between IVPs involving linear and nonlinear ODEs in the following table:

Linear Equations: y' + py = g	**Nonlinear Equations**
A formula exists for the solution, perhaps involving integrals.	Solutions may or may not be expressible explicitly. They may be given implicitly.
We can identify all points of discontinuity of the solution by determining where the coefficient functions p and g are discontinuous.	The points of discontinuity of solutions are more difficult to discern.
A general solution, containing an arbitrary constant, can be written down. Particular solutions can be found by choosing this constant appropriately.	It is not always possible to write down an expression involving an arbitrary constant which encompasses all solutions.