Applications:

Mixing Problems:

- Involve the concentration of a substance in some medium.
  - Flow rate and concentration of drugs in the bloodstream.
  - Concentration of chemicals in a body of water.
- This type of problem is common in many disciplines.
- These problems correspond to interesting differential equations.

To answer applied problems involving differential equations, think about the following:

1. What is the question is asking?

2. What information are you given?

3. What variable assignments should you make that are consistent with the question and given information?

4. What are the relationships between the given rates (derivatives) and the other parameters?

5. Can you use your model to solve the problem?

Example. *Suppose that a tank contains 20 gallons of solution of a certain chemical and that 5 lb of the chemical are dissolved in the solution. A solution containing the same chemical, with a concentration of 2 lb/gallon, is allowed to flow into the tank at a rate of 3 gal/min. The mixture is drained off at the same rate. At what time after the process starts does the tank contain 25 lb of the chemical?*

Variables:
$$t = \text{time},$$
$$x(t) = \text{amount of chemical in tank at time t},$$

With these assignments,

$$\frac{dx}{dt} = \text{rate of change of the chemical in the tank at time } t.$$

Key Relationship Between Parameters:

$$\left(\begin{array}{l} \text{Rate of change of} \\ \text{amnt. of chemical} \end{array} \right) = \left(\text{Rate flowing in} \right) - \left(\text{Rate flowing out} \right)$$

Rate of chemical flowing in:

$$(\text{concentration of inflow}) \times (\text{rate of inflow}) = 2 \times 3 \text{ lb/min}$$

Rate of chemical flowing out:

$$(\text{concentration of outflow}) \times (\text{rate of outflow}) = \frac{x(t)}{20} \times 3 \text{ lb/min}.$$

So we have the differential equation

$$\frac{dx}{dt} = 6 - \frac{3}{20}x.$$

We can solve this using integrating factors. First rewrite as

$$\frac{dx}{dt} + \frac{3}{20}x = 6.$$

The integrating factor is

$$\mu(t) = e^{3t/20}.$$

The general solution is

$$x(t) = 40 + Ce^{-3t/20}.$$

Using the given information about the initial amount of chemical in the tank, $x(0) = 5$ lb, we see that

$$5 = 40 + Ce^0 \qquad \Longrightarrow \qquad C = -35.$$

Therefore, the solution the the IVP is

$$x(t) = 40 - 35e^{-3t/20}.$$

We want to find the value of t when $x(t)$ is 25.

$$\begin{aligned} x(t) = 25 \qquad &\Longrightarrow \qquad 40 - 35e^{-3t/20} = 25 \\ &\Longrightarrow \qquad 15 = 35e^{-3t/20} \\ &\Longrightarrow \qquad \frac{3}{7} = e^{-3t/20} \\ &\Longrightarrow \qquad \ln\left(\frac{3}{7}\right) = -\frac{3t}{20}. \end{aligned}$$

Hence, the tank contains 25 lbs of the chemical after

$$\boxed{t = -\frac{20}{3}\ln\left(\frac{3}{7}\right) = \frac{20}{3}\ln\left(\frac{3}{7}\right)^{-1} = \frac{20}{3}\ln\frac{7}{3} \text{ minutes.}}$$

Remark.

- *How much chemical will accumulate in the tank in the long run?*

  Will the amount of chemical increase without bound?

  Will the amount of chemical in the tank approach some limiting value?

To answer these questions, we use the form of the solution,

$$x(t) = 40 - 35e^{-3t/20},$$

to observe that, as $t \to \infty$, the amount of chemical in the tank approaches

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}(40 - 35e^{-3t/20}) = \lim_{t\to\infty}\left(40 - 35\frac{1}{e^{3t/20}}\right) = 40.$$

- *How much time will it take for the amount of chemical in the tank to reach 3 times its initial amount?*

To answer this question, we need to determine the t for which

$$x(t) = 3x(0) = 3 \cdot 5 = 15.$$

This means that

$$\begin{aligned} 35e^{-3t/20} = 25 \quad &\Longrightarrow \quad e^{-3t/20} = 5/7 \\ &\Longrightarrow \quad -3t/20 = \ln(5/7) \\ &\Longrightarrow \quad t = -\frac{20}{3}\ln(5/7). \end{aligned}$$

Does this answer makes sense? Is this last expression negative?

Why or why not?

Example. *A tank holds 100 gallons of a solution that contains 40 lb of a chemical. A solution containing 2 lb/gal of the chemical runs into the tank at a rate of 2 gal/min. The mixture runs out at a rate of 3 gal/min. How much chemical is in the tank after 50 min?*

Variables:

$$\begin{aligned} t &= \text{time}, \\ y(t) &= \text{amount of chemical in tank at time t}, \end{aligned}$$

The model is

$$\frac{dy}{dt} = 2 \times 2 - \frac{3y}{100 - 3t + 2t} = 4 - \frac{3y}{100 - t}.$$

That is,

$$\frac{dy}{dt} + \left(\frac{3}{100-t}\right) y = 4.$$

Using the method of integrating factors, we derive

$$\begin{aligned}\mu(t) = \exp\left(\int \frac{3}{100-t} dt\right) &= \exp\left(\int \frac{d}{dt}\left\{-3\ln(100-t)\right\} dt\right) \\ &= \exp\Big(-3\ln(100-t)\Big) \\ &= \exp\Big(\ln(100-t)^{-3}\Big) = (100-t)^{-3}.\end{aligned}$$

Multiplying boths sides of the DE

$$\frac{dy}{dt} + \left(\frac{3}{100-t}\right) y = 4$$

by the integration factor, we find that

$$(100-t)^{-3}\frac{dy}{dt} + (100-t)^{-3}\left(\frac{3}{100-t}\right) y = 4(100-t)^{-3}$$

We now simplify this expression to

$$(100-t)^{-3}\frac{dy}{dt} + 3(100-t)^{-4}y = 4(100-t)^{-3},$$

and express the left side as the derivative of the product $\mu y = (100-t)^{-3}y$,

$$\begin{aligned}\frac{d}{dt}\Big((100-t)^{-3}y\Big) = 4(100-t)^{-3} \qquad &\Longrightarrow \qquad (100-t)^{-3}y = \int 4(100-t)^{-3}dt + C \\ &\Longrightarrow \qquad (100-t)^{-3}y = 2(100-t)^{-2} + C.\end{aligned}$$

Solving for y, we find that

$$y(t) = 200 - 2t + C(100-t)^3.$$

The initial condition means that

$$y(0) = 40,$$

and therefore,

$$200 - 2 \cdot 0 + C(100-0)^3 = 40 \qquad \Longrightarrow \qquad 200 + 100^3 C = 40$$

so we derive

$$C = -\frac{160}{100^3}.$$

Therefore, the amount of chemical in the tank at time t is

$$y(t) = 200 - 2t - \frac{160}{100^3}(100-t)^3.$$

We now use this expression to compute the amount of chemical in the tank after 50 minutes (This is what we were asked to do originally):

$$\boxed{y(50) = 200 - 100 - \frac{160}{100^3}50^3.}$$

Example. *Assume that the rate of change of the supply, $S(t)$, of a commodity is proportional to the difference between the demand, $D(t)$, and the supply. Find a formula for $S(t)$ if the demand is constant, i.e., $D(t) = D_0$.*

> We should be careful to distinguish functions of t in our model from constants which do not depend upon t.

The given assumptions imply

$$\frac{dS}{dt} = r(D - S) = r(D_0 - S),$$

where r is some constant of proportionality.

This differential equation can be recast in standard form as

$$\frac{dS}{dt} + rS = rD_0.$$

The integration factor is

$$\mu(t) = \exp\left(\int r \; dt\right) = e^{rt}.$$

Therefore, the differential equation is equivalent to

$$\begin{aligned}
e^{rt}\frac{dS}{dt} + e^{rt}rS = e^{rt}rD_0 \qquad &\Longrightarrow \qquad \frac{d}{dt}(e^{rt}S) = rD_0e^{rt} \\
&\Longrightarrow \qquad e^{rt}S = rD_0\int e^{rt}\ dt + C \\
&\Longrightarrow \qquad e^{rt}S = D_0e^{rt} + C.
\end{aligned}$$

We conclude that the supply function in terms of the demand, D_0, is

$$\boxed{S(t) = D_0 + Ce^{-rt}.}$$

Example. *The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are* $200,000$ *mosquitoes in the area initially, and predators (birds, bats, and so forth) eat* $20,000$ *mosquitoes each day. Determine the population of mosquitoes in the area at time* t.

Let $P(t)$ be the population of mosquitoes (in thousands) at time t (in days).

In the absence of predators, the rate of change of the population is

$$\frac{dP}{dt} = rP.$$

Solving the differential equation for P, and using the initial condition

$$P(0) = 200,$$

we find that

$$P(t) = 200e^{rt}.$$

Remember, this $P(t)$ represents the population without predation.

Since, without predation, the population of mosquitoes doubles each week, we know that

$$P(7) = 2P(0) \qquad \Longrightarrow \qquad 200e^{7r} = 400e^{0r} \qquad \Longrightarrow \qquad r = \frac{1}{7}\ln 2.$$

In the presence of predators, the rate of change of the population is

$$\frac{dP}{dt} = rP - 20.$$

Therefore, using the value for r above, we see that

$$\frac{dP}{dt} = \frac{\ln 2}{7}P - 20 \qquad \Longrightarrow \qquad \frac{dP}{dt} - \frac{\ln 2}{7}P(t) = -20.$$

The $P(t)$ in this model represents the population with predation.

The integration factor is

$$\mu(t) := \exp\left(\int -\frac{\ln 2}{7}\,dt\right) = \exp\left(-\frac{\ln 2}{7}t\right) = \exp\left(-\frac{t}{7}\ln 2\right) = \exp(\ln 2^{-t/7}) = 2^{-t/7}.$$

Now multiply both sides of the DE,

$$\frac{dP}{dt} - \frac{\ln(2)}{7}P(t) = -20,$$

by the integration factor and recognize the left side as the derivative of $\mu y = 2^{-t/7}y$ to derive

$$\frac{d}{dt}(2^{-t/7}P) = -20 \cdot 2^{-t/7}.$$

Therefore,

$$2^{-t/7}P = -20\int 2^{-t/7}dt + C.$$

To evaluate the integral on the right, note that

$$\int 2^{-t/7}dt = \int e^{\frac{-\ln 2}{7}t}dt = \frac{e^{\frac{-\ln 2}{7}t}}{\left(-\frac{\ln 2}{7}\right)} = -\frac{7}{\ln 2}e^{\frac{-\ln 2}{7}t} = -\frac{7}{\ln 2}2^{-t/7}.$$

Therefore,

$$2^{-t/7}P(t) = 140\frac{2^{-t/7}}{\ln 2} + C,$$

or, equivalently,

$$P(t) = \frac{140}{\ln 2} + 2^{t/7}C.$$

Since $P(0) = 200$, we know that

$$C = 200 - \frac{140}{\ln 2}.$$

Therefore,

$$\boxed{P(t) = \frac{140}{\ln 2} + 2^{t/7}\left(200 - \frac{140}{\ln 2}\right).}$$