Example. *Use an integration factor to find the general solution of the equation*

$$ty' + 2y = \sin t, \qquad t > 0.$$

Write this in the form considered at the end of Tuesday's notes:

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \Longleftrightarrow \qquad y' + \frac{2}{t}y = \frac{\sin t}{t}.$$

That is,

$$p(t) = \frac{2}{t}, \qquad g(t) = \frac{\sin t}{t}.$$

Therefore,

$$\mu(t) = \exp\left(\int p(t)\ dt\right) = \exp\left(\int \frac{2}{t}\ dt\right) = e^{2\ln t} = e^{\ln t^2} = t^2.$$

Finally,

$$\begin{aligned} y = \frac{1}{\mu(t)}\int \mu(t)g(t)\ dt + \frac{C}{\mu(t)} &= \frac{1}{t^2}\int t^2\frac{\sin t}{t}\ dt + \frac{C}{t^2} \\ &= \frac{1}{t^2}\int t\sin t\ dt + \frac{C}{t^2} \\ &= \frac{1}{t^2}(-t\cos t + \sin t) + \frac{C}{t^2}. \end{aligned}$$

Remark. *Before solving a differential equation using the method of integrating factors, you should write the equation in the form*

$$\frac{dy}{dt} + yp(t) = g(t).$$

Then you can either apply the formulas from class (i.e., memorize the appropriate formulas), or you can apply the method to derive these formulas in your specific case.

It's better to understand the method and memorize less.

Separable Equations

Definition. *If a first order ODE is of the form*

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0, \tag{1}$$

then that ODE is ***separable*** *if*

$$M(x,y) = M_1(x)M_2(y) \qquad \textit{and} \qquad N(x,y) = N_1(x)N_2(y). \tag{2}$$

Remark. *The "separate" in separable equations means that we can separate $M(x,y)$ and $N(x,y)$ into their factors, each of which depends on only one variable.*

Dividing equation (1) by $N_1(x)M_2(y)$, using (2), and setting

$$M(x) = \frac{M_1(x)}{N_1(x)} \qquad \text{and} \qquad N(y) = \frac{N_2(y)}{M_2(y)},$$

we see that a separable equation can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{3}$$

Equation (3) is sometimes written as

$$M(x)\ dx + N(y)\ dy = 0.$$

This is an abuse of notation, but writing the equation in this way is often helpful in solving the problem.

Example.

$$\frac{dy}{dx} = e^{x+y}.$$

We first express this as

$$-e^x e^y + \frac{dy}{dx} = 0. \tag{4}$$

This equation is separable since, referring back to equation (1),

$$M(x,y) = -e^x e^y, \qquad N(x,y) = 1.$$

We recast (4) as

$$-e^x + e^{-y}\frac{dy}{dx} = 0, \tag{5}$$

and then, through implicit differentiation,

$$-e^x - \frac{d}{dx}(e^{-y}) = 0.$$

Finally, we integrate both sides with respect to x:

$$-e^x - e^{-y} = C, \qquad C \in \mathbb{R}. \tag{6}$$

Then

$$e^{-y} = -C - e^x \qquad \Longrightarrow \qquad y = -\ln(-C - e^x),$$

provided that $-C - e^x > 0$ (or, equivalently, $C < -e^x$).

Remark. *We could have abused notation at equation* (5) *to write*

$$-e^x\ dx + e^{-y}\ dy = 0.$$

Then

$$-e^x\ dx = -e^{-y}\ dy,$$

so that

$$-\int e^x\ dx = -\int e^{-y}\ dy + C,$$

and, consequently,

$$-e^x = e^{-y} + C,$$

which is equivalent to equation (6).

Example. *Solve the initial value problem*

$$\frac{dy}{dx} = \frac{\cos x}{3y^2 + e^y}, \qquad y(0) = 2.$$

We first rewrite the equation as

$$(3y^2 + e^y)\ dy = \cos x\ dx.$$

Integrating, we derive

$$y^3 + e^y = \sin(x) + C. \tag{7}$$

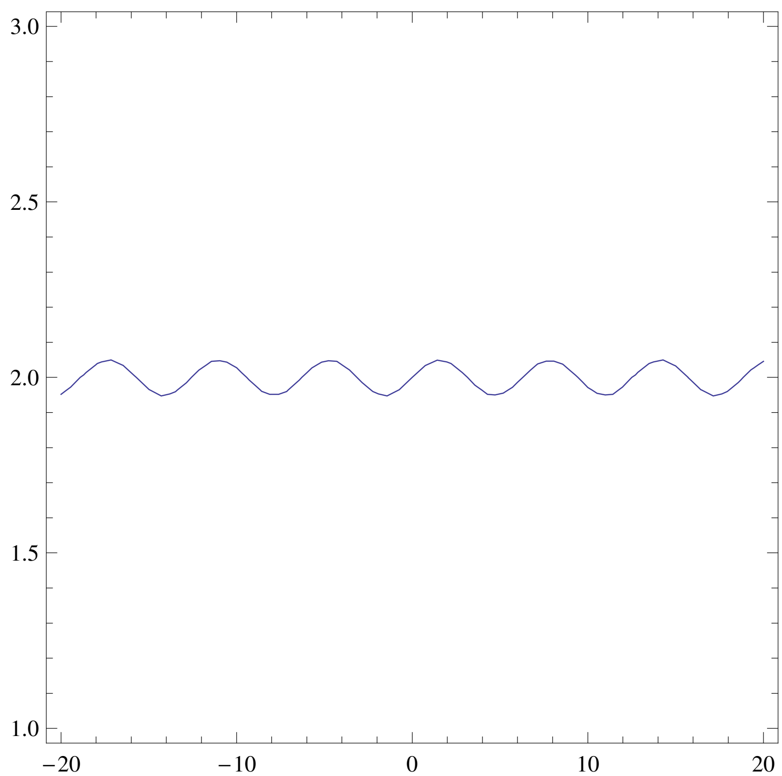
From the initial condition,

$$2^3 + e^2 = \sin 0 + C \qquad \Longrightarrow \qquad C = 8 + e^2.$$

Therefore, from equation (7), we have

$$y^3 + e^y = \sin(x) + 8 + e^2.$$

Here y is given implicitly as a function of x (and this is the best we can do).



Example. *Find the general solution to the ODE*

$$3y^2 y' = (1 + y^3)\cos x.$$

First write the equation as

$$\frac{3y^2}{1+y^3}\frac{dy}{dx} = \cos x, \qquad y \neq -1.$$

So we find

$$\frac{3y^2}{1+y^3}\, dy = \cos x\ dx,$$

and then, by anti-differentiating

$$\begin{aligned}
\ln|1+y^3| = \sin x + C \qquad &\Longrightarrow \qquad |1+y^3| = Ke^{\sin x}, \qquad K > 0,\\
&\Longrightarrow \qquad 1 + y^3 = Ke^{\sin x}, \qquad K \neq 0,\\
&\Longrightarrow \qquad y = (Ke^{\sin x} - 1)^{1/3}, \qquad K \neq 0.
\end{aligned}$$

We extend K to also be zero because it can be easily verified that

$$y = (-1)^{1/3} = -1$$

is a solution to the differential equation.

Therefore, the general solution to the differential equation is

$$y = (Ke^{\sin x} - 1)^{1/3}, \qquad K \in \mathbb{R}.$$

Example. *Solve the differential equation*

$$xy' = (1 - y^2)^{1/2}.$$

First rewrite the equation as

$$(1-y^2)^{-1/2}\frac{dy}{dx} = \frac{1}{x}, \qquad y \neq \pm 1$$

so that

$$(1 - y^2)^{-1/2}\ dy = \frac{1}{x}\ dx, \qquad x \neq 0.$$

By integrating this expression, we find

$$\arcsin(y) = \ln|x| + C, \qquad -1 < y < 1.$$

Taking the sine of both sides,

$$y = \sin(\ln|x| + C), \qquad -1 < y < 1.$$

You can also check that $y = \pm 1$ are also solutions to the differential equation.