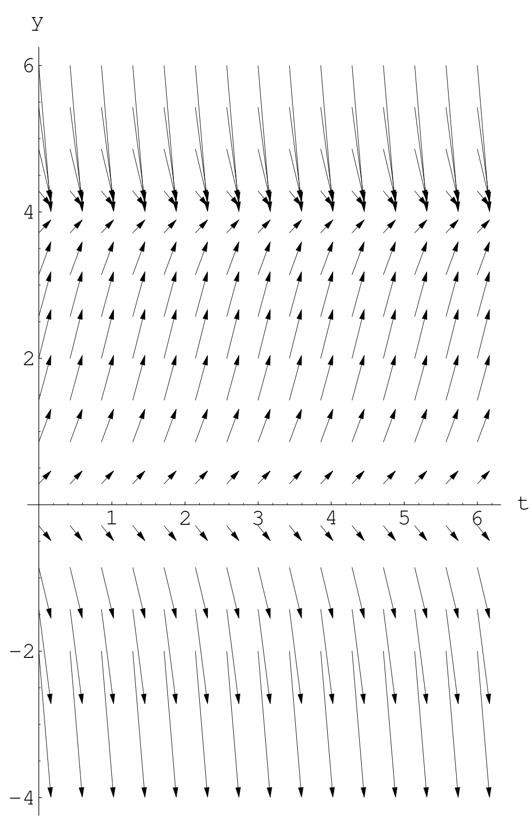
Remarks.

- *A comparatively small collection of differential equations can be solved in "closed form" using simple functions.*
- *In this course, we will look at methods for finding solutions, and, when this is not feasible (or possible), we will examine qualitative methods to study solutions of differential equations.*

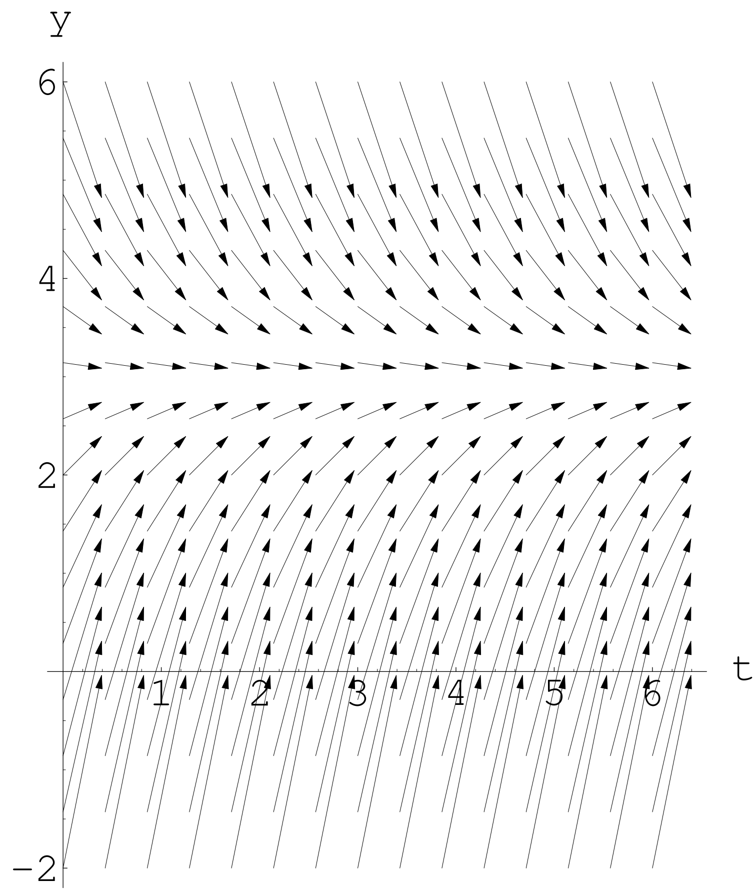
An important qualitative analysis tool for studying a differential equation is a **direction/slope field**

Definition. *A* ***direction field*** *or* ***slope field*** *is a collection of vectors which are tangent to the graph of solutions to a differential equation.*

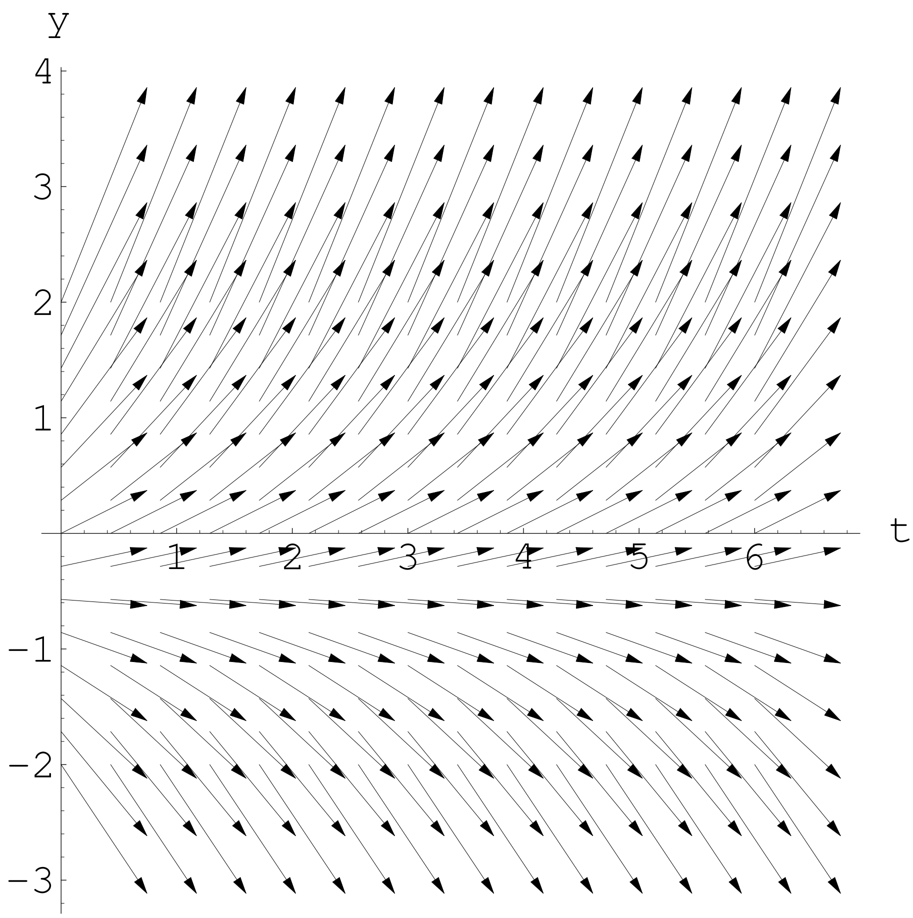
Example. *Direction field for* $\dfrac{dy}{dt} = y(4-y)$



Example. *Direction field for* $\dfrac{dy}{dt} = 3 - y$



Example. *Direction field for* $\dfrac{dy}{dt} = \dfrac{1}{2} + y$



Definition. *A solution to a differential equation* $\frac{dy}{dt} = f(y,t)$ *corresponding to* $\frac{dy}{dt} = 0$ *is called an* ***equilibrium solution****. Solutions may converge to or diverge from equilibrium solutions depending on initial values.*

In general, to find equilibrium solutions, set $\frac{dy}{dt} = 0$ in the differential equation and solve for y.

Question: How do we tell if solutions asymptotically converge/diverge to/from an equilibrium solution? **Answer:** This can be determined from the sign of the derivative above/below an equilibrium solution.

Example. *Find the equilibrium solutions of the differential equation*

$$\frac{dy}{dt} = ay + b,$$

where a and b are constants.

These occur at

$$0 = \frac{dy}{dt} = ay + b,$$

so that

$$y = -\frac{b}{a}.$$

Example. *Find all solutions to the differential equation*

$$\frac{dy}{dt} = ay + b, \qquad y(0) = y_0,$$

where $a, b,$ and y_0 are fixed constants.

We first recast the equation:

$$\frac{dy}{dt} = ay + b \qquad \Longrightarrow \qquad \frac{\frac{dy}{dt}}{ay + b} = 1 \qquad \Longrightarrow \qquad \frac{\frac{dy}{dt}}{y + b/a} = a.$$

Next, integrate both sides with respect to t to derive

$$\ln |y + b/a| = at + K,$$

where K is the constant of integration.

Now exponentiate both sides to obtain

$$|y + b/a| = Le^{at},$$

where $L = e^K$.

$$y + b/a = \pm Le^{at} \qquad \Longrightarrow \qquad y = -b/a + Ce^{at},$$

where $C = \pm L$ represents some constant.

We finally apply the initial condition $y(0) = y_0$ to find that

$$y_0 = -b/a + Ce^{a \cdot 0} = -b/a + C.$$

Therefore,

$$C = y_0 + b/a.$$

We conclude that the solution to the IVP

$$\frac{dy}{dt} = ay + b, \qquad y(0) = y_0,$$

is

$$y = -\frac{b}{a} + \left(y_0 + \frac{b}{a}\right)e^{at}.$$

The Method of Integrating Factors

In the last example, we solved a differential equation by rewriting it and integrating both sides.

What do we do if this is not possible?

For *first order linear equations*, we can try to apply the **method of integrating factors**.

Key Idea: Multiply both sides of the equation by a function so that one side takes the form of the product rule:

$$(fg)' = f'g + g'f.$$

The difficulty lies in finding this function.

Example. *Solve the initial value problem*

$$y' - 2y = e^{2t}, \qquad y(0) = 2.$$

First multiply both sides by some unknown function $\mu(t)$, the ***integration factor****:*

$$\mu(t)y' - 2\mu(t)y = e^{2t}\mu(t). \tag{1}$$

The product rule says that

$$\frac{d}{dt}\Big(\mu(t)y\Big) = \mu(t)y' + y\mu'(t). \tag{2}$$

Choose $\mu(t)$ so that the right side of (2) matches the left side of (1).

We're set if

$$\mu'(t) = -2\mu(t)$$

This means that

$$\begin{aligned}
\mu'(t) = -2\mu(t) &\Longrightarrow \frac{\mu'(t)}{\mu(t)} = -2 \qquad (\textit{provided } \mu(t) > 0) \\
&\Longrightarrow \frac{d}{dt}\ln\mu(t) = -2 \qquad (\textit{since } \tfrac{d}{dt}\ln\mu(t) = \tfrac{\mu'(t)}{\mu(t)}) \\
&\Longrightarrow \ln\mu(t) = -2t + C \qquad (\textit{integrate}) \\
&\Longrightarrow \mu(t) = Ke^{-2t} \qquad (\textit{exponentiate, where } K = e^C)
\end{aligned}$$

Since $\mu(t)$ is an integration factor, we do not need the most general solution, $\mu(t)$ (we just need a function $\mu(t)$ that works). We may take

$$\mu(t) = e^{-2t}.$$

Plugging this back into (1) we obtain

$$\begin{aligned}
& e^{-2t}y' - 2ye^{-2t} = 1 \\
&\Longrightarrow \frac{d}{dt}(e^{-2t}y) = 1 \qquad (\textit{rewrite the left side}) \\
&\Longrightarrow e^{-2t}y = t + C \qquad (\textit{integrate}) \\
&\Longrightarrow y = te^{2t} + Ce^{2t} \qquad (\textit{solve for } y).
\end{aligned}$$

Now we use the initial condition $y(0) = 2$ *and the general solution above to conclude that*

$$2 = y(0) = 0 \cdot e^{2\cdot 0} + Ce^{2\cdot 0} = C.$$

Therefore, the solution to the inital value problem is

$$y = te^{2t} + 2e^{2t}.$$

In the last example, we found $\mu(t)$ by solving a differential equation. How do we solve for $\mu(t)$ in general (if this is even possible)?

The most general first order linear ODE can be written in the form

$$a_0(t) + a_1(t)y + a_2(t)\frac{dy}{dt} = 0. \tag{3}$$

If we set

$$g(t) = -\frac{a_0(t)}{a_2(t)} \qquad \text{and} \qquad p(t) = \frac{a_1(t)}{a_2(t)},$$

then (3) can be written in the form

$$\frac{dy}{dt} + p(t)y = g(t).$$

Multiply both sides by an integration factor $\mu(t)$ to obtain

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t). \tag{4}$$

We want $\mu(t)$ so that the left side equals

$$\frac{d}{dt}\Big(\mu(t)y\Big).$$

This means that

$$\begin{aligned}
\frac{d\mu(t)}{dt} = p(t)\mu(t) \qquad &\Longrightarrow \qquad \frac{\mu'(t)}{\mu(t)} = p(t) \qquad (\text{provided } \mu(t) > 0)\\
&\Longrightarrow \qquad \frac{d}{dt}\ln\mu(t) = p(t) \qquad (\text{derivative rule for } \ln f(t))\\
&\Longrightarrow \qquad \ln\mu(t) = \int p(t)dt + C \qquad (\text{integrate})\\
&\Longrightarrow \qquad \mu(t) = C\exp\Big(\int p(t)dt\Big) \qquad (\text{exponentiate})
\end{aligned}$$

Therefore, we may take our integration factor to be

$$\mu(t) = \exp\Big(\int p(t)dt\Big). \tag{5}$$

So, if we can compute this integral, we will obtain $\mu(t)$ in closed form.

We can then rewrite (4) as

$$\frac{d}{dt}\Big(y \cdot \mu(t)\Big) = \mu(t)g(t).$$

Integrating this expression, we obtain

$$y \cdot \mu(t) = \int \mu(t)g(t)dt + C.$$

Finally, solving for y,

$$y = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + \frac{C}{\mu(t)}. \tag{6}$$

Therefore, we have solved the most general first order linear differential equation, provided we can compute the integrals appearing in (5) and (6).

Unfortunately, this is not always possible.