Complex Roots of the Characteristic Equation for

$$ay'' + by' + cy = 0,$$

We know how to solve

$$ay'' + by' + cy = 0.$$

when the characteristic equation

$$ar^2 + br + c = 0$$

has real distinct roots r_1 and r_2. In this case, the general solution is given by

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

What do we do when $b^2 - 4ac$ is a negative number?

That is, how do we interpret the solutions

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad ?$$

In this case what do we mean by

$$e^{r_1 t} \qquad \text{and} \qquad e^{r_2 t}$$

when r_1 and r_2 involve the square roots of negative numbers?

And, by the way, what the heck is a complex number?

We will denote the square root of -1 as i. In other words, we define i, formally, to be the number such that

$$i \cdot i = -1.$$

A **complex number** is a number of the form $\lambda + \mu i$, for any real numbers λ and μ.

Here are some examples of complex numbers:

$$3.7 + 5.9i, \qquad \pi + 6i, \qquad e + \frac{i}{2}.$$

We represent the square roots of negative numbers using the following convention:

For any positive real number μ,

$$\sqrt{-\mu} = i\sqrt{\mu}.$$

For example,

$$\sqrt{-10} = i\sqrt{10}, \qquad \sqrt{-25} = 5i, \qquad 1 - \frac{\sqrt{-5}}{2} = 1 - \frac{\sqrt{5}}{2}i.$$

Next, we define what we mean when we exponentiate the product of a real number and i.

Euler's Formula: For any real number t,

$$e^{it} = \cos t + i \sin t.$$

Leonhard Euler (1707 - 1783)

For example,

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i \cdot 0 = 1,$$

$$e^{\pi i/3} = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$e^{\frac{\pi}{2} i} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i \cdot 1 = i,$$

$$e^{(\log 3) i} = \cos(\log 3) + i \sin(\log 3),$$

and

$$e^{-it} = \cos(-t) + i \sin(-t) = \cos(t) - i \sin(t).$$

This last equality follows from the trigonometric identities

$$\cos(-x) = \cos(x) \qquad \text{and} \qquad \sin(-x) = -\sin(x).$$

Euler's Formula comes from an evaluation of the infinite series representation for the exponential function

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t.$$

It now makes sense, with these conventions, to write

$$e^{\lambda+\mu i}, \qquad \text{for real numbers } \lambda \text{ and } \mu.$$

Using laws of exponents, we get, for real numbers λ and μ,

$$e^{\lambda+\mu i} = e^{\lambda} e^{\mu i} = e^{\lambda}(\cos\mu + i\sin\mu).$$

Let's return to the differential equation

$$ay'' + by' + cy = 0$$

in the case that $b^2 - 4ac$ is a negative number.

Theorem. *The general solution to*

$$ay'' + by' + cy = 0$$

when

$$b^2 - 4ac$$

is negative is

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t,$$

where

$$\lambda = -\frac{b}{2a} \qquad \textit{and} \qquad \mu = \frac{\sqrt{|b^2-4ac|}}{2a}.$$

Proof. All of the derivative rules, including

$$\frac{d}{dt}e^{rt} = re^{t}$$

work for complex, as well as real, values of r. Therefore, to solve

$$ay'' + by' + cy = 0$$

when $b^2 - 4ac$ is negative, we solve as we would in the case $b^2 - 4ac > 0$. We get two solutions

$$y_1(t) = e^{r_1 t}, \qquad y_2(t) = e^{r_2 t},$$

where r_1 and r_2 are roots of the characteristic equation

$$ar^2 + br + c = 0,$$

so that

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Since $b^2 - 4ac$ is negative,

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{|b^2 - 4ac|}}{2a}i, \qquad r_2 = -\frac{b}{2a} - \frac{\sqrt{|b^2 - 4ac|}}{2a}i.$$

If we set, from now on,

$$\lambda = -\frac{b}{2a} \qquad \text{and} \qquad \mu = \frac{\sqrt{|b^2 - 4ac|}}{2a},$$

then

$$r_1 = \lambda + \mu i \qquad \text{and} \qquad r_2 = \lambda - \mu i.$$

Therefore, with this notation, our solutions become

$$y_1(t) = e^{(\lambda+\mu i)t} \qquad \text{and} \qquad y_2(t) = e^{(\lambda-\mu i)t}.$$

Using Euler's Formula, we find that

$$\begin{aligned} y_1(t) + y_2(t) &= e^{\lambda t} e^{\mu t i} + e^{\lambda t} e^{-\mu t i} \\ &= e^{\lambda t}(\cos \mu t + i \sin \mu t) + e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t + e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t \\ &= 2e^{\lambda t} \cos \mu t. \end{aligned}$$

Likewise,

$$\begin{aligned} y_1(t) - y_2(t) &= e^{\lambda t} e^{\mu t i} - e^{\lambda t} e^{-\mu t i} \\ &= e^{\lambda t}(\cos \mu t + i \sin \mu t) - e^{\lambda t}(\cos \mu t - i \sin \mu t) \\ &= e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t - e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t \\ &= 2i e^{\lambda t} \sin \mu t. \end{aligned}$$

Since y_1 and y_2 are solutions to the differential equation, so are

$$u(t) = \frac{1}{2}(y_1(t) + y_2(t)) = e^{\lambda t} \cos \mu t$$

and

$$v(t) = \frac{1}{2i}(y_1(t) - y_2(t)) = e^{\lambda t} \sin \mu t.$$

We can show that

$$W(u, v)(t) = \mu e^{2\lambda t} \neq 0,$$

since

$$b^2 - 4ac < 0 \qquad \Longrightarrow \qquad \mu := \frac{\sqrt{|b^2 - 4ac|}}{2a} \neq 0.$$

Since the Wronskian $W(u,v)(t)$ is nonzero, the functions

$$u(t) = e^{\lambda t}\cos\mu t \qquad \text{and} \qquad v(t) = e^{\lambda t}\sin\mu t$$

form a fundamental set of solutions. This is the same as saying that the general solution to the differential equation is

$$y(t) = c_1 u(t) + c_2 v(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t.$$

This is what we wanted to prove. □

So, how do we apply the theorem in practice, without having to memorize too much?

Solving $ay'' + by' + cy = 0$ when the roots of the characteristic equation are complex:

1. Find the complex roots, r_1, r_2 to $ar^2 + br + c = 0$

$$r_1 = \lambda + \mu i,$$
$$r_2 = \lambda - \mu i.$$

2. Use Euler's formula to write out one solution in terms of trigonometric functions:

$$\begin{aligned} y(t) = e^{(\lambda+\mu i)t} = e^{\lambda t+\mu t i} &= e^{\lambda t}e^{\mu t i} \\ &= e^{\lambda t}(\cos\mu t + i\sin\mu t) \\ &= e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t. \end{aligned}$$

3. One solution to the differential equation is the **real part** of $y(t)$ and the other solution is the **imaginary part** of $y(t)$:

$$\text{Real part of } y(t) = e^{\lambda t}\cos\mu t,$$
$$\text{Imaginary part of } y(t) = e^{\lambda t}\sin\mu t.$$

Example. *Find the general solution to*

$$y'' - 2y' + 6y = 0.$$

The roots of the characteristic equation

$$r^2 - 2r + 6 = 0$$

are

$$r = \frac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm \frac{\sqrt{20}}{2} i = 1 \pm \sqrt{5} i.$$

In other words, $r = \lambda \pm \mu i$, where

$$\lambda = 1 \qquad \text{and} \qquad \mu = \sqrt{5}.$$

From the theorem above, the general solution is

$$\begin{aligned} y(t) &= c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \\ &= c_1 e^t \cos \sqrt{5} t + c_2 e^t \sin \sqrt{5} t. \end{aligned}$$

Example. *Solve the initial value problem*

$$y'' + 2y' + 2y = 0, \qquad y(\pi/4) = 2, \qquad y'(\pi/4) = -2.$$

Our differential equation is of the form

$$ay'' + by' + cy = 0,$$

where

$$a = 1, \qquad b = 2, \qquad c = 2.$$

Thus, the roots of the characteristic equation are

$$r = \frac{-2 \pm \sqrt{-4}}{2},$$

so that

$$r = \lambda \pm \mu i,$$

where

$$\lambda = -\frac{b}{2a} = -1 \qquad \text{and} \qquad \mu = \frac{\sqrt{|b^2 - 4ac|}}{2a} = 1.$$

By the theorem, the general solution to this differential equation is

$$\begin{aligned} y(t) &= c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t \\ &= c_1 e^{-t} \cos t + c_2 e^{-t} \sin t. \end{aligned}$$

We need to find constants, c_1 and c_2, so that

$$y(\pi/4) = 2 \qquad \text{and} \qquad y'(\pi/4) = -2.$$

Since

$$y(t) = e^{-t}(c_1 \cos t + c_2 \sin t),$$

we apply the product rule to calculate

$$y'(t) = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-c_1 \sin t + c_2 \cos t).$$

Therefore, using $\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we rewrite our initial conditions as

$$2 = y(\pi/4) = e^{-\pi/4}\left(c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}\right) = e^{-\pi/4}\left(\frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}\right), \tag{1}$$

and, taking great care,

$$\begin{aligned} -2 = y'(\pi/4) &= -e^{-\pi/4}(c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}) \\ &\qquad + e^{-t}(-c_1 \sin \frac{\pi}{4} + c_2 \cos \frac{\pi}{4}) \\ &= -e^{-\pi/4}(c_1 \frac{1}{\sqrt{2}} + c_2 \frac{1}{\sqrt{2}}) \\ &\qquad + e^{-\pi/4}(-c_1 \frac{1}{\sqrt{2}} + c_2 \frac{1}{\sqrt{2}}) \\ &= -\frac{2}{\sqrt{2}} e^{-\pi/4} c_1. = -\sqrt{2} e^{-\pi/4} c_1 \end{aligned} \tag{2}$$

Equation (2) implies that

$$c_1 = \frac{2}{\sqrt{2}} e^{\pi/4} = \sqrt{2} e^{\pi/4}.$$

Using this value for c_1 in equation (1) implies that

$$2\sqrt{2} e^{\pi/4} = \sqrt{2} e^{\pi/4} + c_2 \qquad \Longrightarrow \qquad c_2 = \sqrt{2} e^{\pi/4}.$$

Therefore, the solution is given by

$$y(t) = \sqrt{2} e^{\pi/4} e^{-t} \cos t + \sqrt{2} e^{\pi/4} e^{-t} \sin t.$$