Linear Independence and the Wronskian

Two functions $f(t)$ and $g(t)$ are said to be linearly dependent on an interval I if there are two constants k_1 and k_2, not both zero, so that

$$k_1 f(t) + k_2 g(t) = 0$$

for all t in I.

Two functions are linearly independent if they are not linearly dependent.

We can extend these definitions to any number of functions.

The n functions $f_1(t), f_2(t), \ldots, f_n(t)$ are said to be linearly dependent on an interval I if there exist n constants $k_1, k_2, \ldots, k_n$, not all zero, so that

$$k_1 f_1(t) + k_2 f_2(t) + \cdots + k_n f_n(t) = 0$$

for all t in I.

Otherwise, the n functions are linearly independent.

Example. *Are the functions*

$$f(t) = 5, \qquad g(t) = \sin^2(t) + \cos^2(t)$$

linearly independent or linearly dependent on $(-\infty, \infty)$*?*

These functions are linearly dependent since, by choosing

$$k_1 = \frac{1}{5}, \qquad \text{and} \qquad k_2 = -1,$$

we find, from the identity

$$\sin^2(t) + \cos^2(t) = 1 \qquad \text{for all } t \in (-\infty, \infty),$$

that

$$k_1 f(t) + k_2 g(t) = 1 - \sin^2(t) - \cos^2(t) = 0$$

for all t in $(-\infty, \infty)$.

We need the following theorem before we continue.

Theorem. *Let a, b, c, d be constants. The system of equations*

$$ax + by = 0$$
$$cx + dy = 0$$

has, as the only solution

$$x = 0, \qquad y = 0$$

if and only if the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$$

Example. *Are the two functions $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ linearly dependent or independent on the real line?*

Let's suppose these functions are linearly dependent. That is, there exist two numbers k_1 and k_2, not both zero, so that

$$k_1 e^{-t} + k_2 e^{-2t} = 0, \qquad \text{for all } t \in (-\infty, \infty).$$

Since this works for any real t, we may plug in $t = 1$ and $t = 0$ to get, respectively,

$$\begin{aligned} e^{-1}k_1 + e^{-2}k_2 &= 0 \\ e^{0}k_1 + e^{0}k_2 &= 0. \end{aligned}$$

This coefficients in this system correspond to a matrix with determinant

$$\begin{vmatrix} e^{-1} & e^{-2} \\ e^{0} & e^{0} \end{vmatrix} = (e^{-1} \cdot 1) - (1 \cdot e^{-2}) = e^{-1} - e^{-2} \neq 0.$$

By the theorem above, this system of equations, with unknowns k_1 and k_2, has as its only solution

$$k_1 = 0 \qquad \text{and} \qquad k_2 = 0.$$

This contradicts our assumption that k_1 and k_2 were nonzero.

Therefore, the functions $f(t) = e^{-t}$ and $g(t) = e^{-2t}$ are linearly independent on the real line

The following theorem gives us a way to test for linear independence of two functions on an interval I.

Theorem. *If f and g are differentiable functions on an open interval I, and if $W(f,g)(t_0) \neq 0$ for some point t_0 in I, then f and g are linearly independent on I.*

Example.

Consider the functions in last example

$$f(t) = e^{-t} \qquad \text{and} \qquad g(t) = e^{-2t}.$$

We see, since $f'(t) = -e^{-t}$ and $g'(t) = -2e^{-2t}$, that

$$W(f,g)(0) = \begin{vmatrix} e^0 & e^0 \\ -e^0 & -2e^0 \end{vmatrix} = -2 + 1 = -1 \neq 0.$$

Applying the theorem here, we conclude that

$$f(t) = e^{-t} \qquad \text{and} \qquad g(t) = e^{-2t}$$

are linearly independent on any open interval, I, containing the point $t = 0$.

Therefore, we have shown, using the Wronskian, that $f(t)$ and $g(t)$ are linearly independent on $I = (-\infty, \infty)$.

If we know two functions are linearly dependent on an interval I, this tells us something about the Wronskian of the two functions.

Theorem. *Suppose f and g are differentiable functions on an open interval I. If f and g are linearly dependent on I, then*

$$W(f,g)(t) = 0 \qquad \text{for every } t \text{ in } I.$$

Example. *The trigonometric indentity*

$$\sin t = \cos\left(t - \frac{\pi}{2}\right)$$

implies that

$$\sin t - \cos\left(t - \frac{\pi}{2}\right) = 0 \qquad \text{for all } t \in (-\infty, \infty).$$

This means that the functions

$$f(t) = \sin(t) \qquad \text{and} \qquad g(t) = \cos\left(t - \frac{\pi}{2}\right)$$

are linearly dependent. Evaluate the Wronskian

$$W(f,g)(t)$$

at every point in $\mathbb{R}$.

Since we know that f and g are linearly dependent, the theorem tells us that

$$W(f,g)(t) = 0 \qquad \text{for all } t \in \mathbb{R}.$$

We can compute the Wronskian of two solutions to a differential equation without actually solving the equation.

Theorem. *If y_1 and y_2 are solutions of the differential equation*

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)\ dt\right),$$

where the constant c depends upon y_1 and y_2, but not on t.

This result is sometimes called **Abel's Theorem**.

As a consequence of the theorem, we have the following corollary.

Corollary. *If y_1 and y_2 are solutions of the differential equation*

$$y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is either zero for all t in I (this happens only when $c = 0$) or else it is never zero in I (this happens only when $c \neq 0$).

The basic ideas for the proof of Abel's theorem are simple.

Proof. If y_1 and y_2 both satisfy the differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

we know that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0, \tag{1}$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0. \tag{2}$$

Multiply equation (1) by $-y_2$ and equation (2) by y_1 and add the resulting equations to get

$$\underbrace{(y_1 y_2'' - y_1'' y_2)}_{\substack{= W' \\ \text{(product rule)}}} + p(t) \underbrace{(y_1 y_2' - y_1' y_2)}_{\substack{= W \\ = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}} = 0. \tag{3}$$

So, abbreviating $W(y_1, y_2)(t)$ as W, we can rewrite (3) as

$$\begin{aligned} W' + p(t)W = 0 \quad &\Longrightarrow \quad \frac{W'}{W} = -p(t) \\ &\Longrightarrow \quad \frac{d}{dt} \ln W = -p(t) \\ &\Longrightarrow \quad \ln W = -\int p(t)\ dt + K \\ &\Longrightarrow \quad W = C \exp\left(-\int p(t)\ dt\right), \end{aligned}$$

where $C = e^K$. □

Example. *Find the Wronskian of two solutions on the interval $(0, \pi/2)$ to the differential equation*

$$(\cos t)y'' + (\sin t)y' - ty = 0$$

without solving the differential equation.

The first thing we need to do is to write the differential equation in the form appearing in Abel's Theorem:

$$y'' + \frac{\sin t}{\cos t}y' - \frac{t}{\cos t}y = 0.$$

That is,

$$y'' + (\tan t)y' - \frac{t}{\cos t}y = 0.$$

Since the coefficient functions in this last equation are continuous on $(0, \pi/2)$, Abel's Theorem says that the Wronskian of two solutions y_1, y_2, of this differential equation equals, for some c,

$$\begin{aligned} W(y_1, y_2)(t) &= c\exp\left(-\int p(t)\ dt\right) \\ &= c\exp\left(-\int \tan t\ dt\right) \\ &= c\exp\Big(-(-\ln|\cos t|)\Big) \\ &= c\exp(\ln|\cos t|) \\ &= c|\cos t| = c \cdot \cos t, \end{aligned}$$

where the last equality follows from the fact that

$$\cos t > 0 \qquad \text{for} \qquad t \in (0, \pi/2).$$

If we know that y_1 and y_2 are solutions of a second order linear (homogeneous) differential equation, then linear independence of y_1 and y_2 is equivalent to a number of other useful results.

Theorem. *Let y_1 and y_2 be solutions of the differential equation*

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are continuous on the interval I. Then each the following statements implies any of the others:

1. *The functions y_1 and y_2 form a fundamental set of solutions for the differential equation on I.*

2. *The functions y_1 and y_2 are linearly independent on I.*

3. *The Wronskian $W(y_1, y_2)(t_0) \neq 0$ for some t_0 in I.*

4. *The Wronskian $W(y_1, y_2)(t) \neq 0$ for all t in I.*

Abel's Theorem implies the equivalence of the third and fourth statements.

Complete proofs for each implication are given in sections §3.2 and §3.3.