Course Content

What you should get from this course:

- Acquire the means to solve a variety of differential equations using standard methods.
- Gain an understanding of the basic theory, including key theorems, in the theory of ODEs.
- Develop a familiarization with the technology used to solve and study ODEs.
- Gain experience and build intuition with modeling physical processes using ODEs.

Remarks on the course

- In the past, a typical course in ODEs involved exploring a large number of "recipes" for solving differential equations. Computer algebra systems today automate this process and are used in practice to eliminate a lot of tedium.
- Some of assigned problems require [*or will be significantly simplified through*] the use of a computer-algebra system (e.g., Mathematica, Maple, Matlab, ODE Architect [on the CD in your book]).
- Many of the problems can be solved easily using a computer. However, to understand the underlying ideas and build confidence, you should solve basic problems on paper by hand. Be careful not to rely too much on the computer to do your thinking. Use the computer primarily for repetitive and highly complex problems. Unless specifically stated, you should show all your work in solving a problem. For example, a computer generated solution to a homework problem without theoretical justification will result in zero credit.

§1.1–1.3: Introduction

Definition. *A* ***differential equation*** *is an equation containing derivatives.*

Definition. *An* ***ordinary differential equation*** *(ODE) is a differential equation containing a single independent variable. That is, an ODE is a differential equation that relates a function of one variable to its derivatives.*

Example. *Some physical processes have the property that the rate of change (i.e., the derivative) is proportional to the function itself:*

- *Population dynamics (the rate of increase in population depends upon the number of organisms present).*
- *Radioactive decay (the amount of decay depends upon the amount present at any given time).*

These processes are modeled roughly by the equation

$$\frac{dy}{dt} = ry,$$

where $y := y(t)$ represents the population size (or amount of substance present) at time t.

Applications of differential equations appear in almost every scientific field:

- Physics,
- Chemistry,
- Biology,
- Engineering,
- Astronomy,
- Economics,
- and connections exist with many areas of pure mathematics.

Terminology. *A differential equation that describes some applied dynamic process is called a* <u>*mathematical model*</u>*.*

In contrast to an ODE, a differential equation involving more than one independent variable is called a partial differential equation (PDE). PDEs employ *partial derivatives.*

The focus in this course will be on ODEs.

An ODE can involve more than one function and more than one equation.

Definition. *A set of equations where more than one unknown function is to be determined from more than one equation is called a* ***system of equations****.*

Definition. *The* ***order*** *of a differential equation is the order of the highest derivative appearing in the equation.*

Definition. *A* ***linear differential equation*** *is an equation which is a linear function of the unknown function and its derivatives. In other words, a linear with independent variable t and dependent variable y is one that can be written i the form*

$$a(t) + a_0(t)y + a_1 y(t) + \cdots + a_n(t)y^{(n)}(t) = 0.$$

An equation is nonlinear if it cannot be written in this form.

Example (Linear and Nonlinear Equations)**.**

- *Linear*

$$x(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{x}{4} = 0 \qquad (\textit{second order linear}),$$
$$y' = 3t \qquad (\textit{first order linear}),$$
$$\frac{d^3\theta}{dt^3} = 2\theta \sin t \qquad (\textit{third order linear}).$$

- *Nonlinear*

$$\frac{d^3y}{dx^3} = 3 + y\frac{dy}{dx} \qquad (\textit{third order nonlinear}),$$
$$y' = 3y^2 \qquad (\textit{first order nonlinear}),$$
$$\frac{d^2\theta}{dt^2} + 2\sin\theta = 0 \qquad (\textit{second order nonlinear}).$$

Definition. *A **solution** to a differential equation is a function that satisfies the differential equation.*

Example (Solutions to ODEs)**.**

- $ty' - y = t^2$ *has a solution given by*

$$y = 3t + t^2.$$

- $t^2y'' + 5ty' + 4y = 0,\ t > 0$ *has a solution given by*

$$y = t^{-2}$$

  and by

$$y = t^{-2}\ln t.$$

- *For the differential equation*

$$\frac{dy}{dt} = ry,$$

  the solution is

$$y(t) = Ke^{rt}, \qquad \text{for some } K.$$

  What is the significance of K in this example? Suppose that we seek a solution to the differential equation with the extra condition that

$$y(0) = 45.$$

  This means that

$$y(0) = Ke^{r\cdot 0} = Ke^0 = K.$$

  *Therefore, K represents the value of the function $y(t)$ at $t = 0$. This extra condition is called an **initial condition**.*

  Definition. *A differential equation together with an initial condition is called an **initial value problem**.*

  Definition. *When the initial condition is not specified, we write down all solutions through the use of constants. Such an expression is called the **general solution** to an ODE.*