Homogeneous Linear Systems with Constant Coefficients (Complex Eigenvalues)

The behavior of solutions to the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is determined by the eigenvalues for the matrix $\mathbf{A}$.

If we assume that $\mathbf{A}$ has real entries, then there are three possibilities:

1. All eigenvalues are distinct and real.
2. The eigenvalues occur in complex conjugate pairs.
3. Some eigenvalues are repeated.

In the last lecture we looked at systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in which the eigenvalues of $\mathbf{A}$ were distinct and real.

Now we look at systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in which $\mathbf{A}$ has complex eigenvalues.

Theorem. *If $\lambda = \alpha + i\beta$, $\beta \neq 0$ is a complex eigenvalue of the real matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{x}$, then $\overline{\lambda} = \alpha - i\beta$ is also an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\overline{\mathbf{x}}$.*

Proof. If A is a real-valued matrix, $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial with real coefficients, so complex roots must occur in conjugate pairs. Since A has real entries, $\overline{\mathbf{A}} = \mathbf{A}$, so

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \qquad \Longrightarrow \qquad \mathbf{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}.$$

□

Since it can be difficult to visualize complex-valued vector solutions to the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, we convert solutions involving complex entries to solutions with all real entries.

We find the real-valued vector solutions by taking the real and imaginary parts of the complex solutions.

Example. *Solve the differential equation* $\mathbf{x}' = \mathbf{A}\mathbf{x}$*, with*

$$A = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}.$$

We compute the eigenvalues by finding the zeros of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 5.$$

By the quadratic formula,

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i.$$

Since the eigenvalues are complex conjugates of each other (and there are only two), we can compute one eigenvalue/eigenvector pair, and get the other pair by conjugation.

For each eigenvalue λ, the corresponding eigenvector $\mathbf{x}$ needs to satisfy

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

For $\lambda = -2 + i$, we find that $\mathbf{x}$ must satisfy

$$\begin{pmatrix} -3-(-2+i) & -1 \\ 2 & -1-(-2+i) \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

Row reducing the augmented matrix $(\mathbf{A} - \lambda \mathbf{I} \mid \mathbf{0})$ results in

$$\begin{pmatrix} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1-i & 0 \\ -1-i & -1 & 0 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & \frac{1-i}{2} & 0 \\ -1-i & -1 & 0 \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & \frac{1-i}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the eigenvector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfies

$$x_1 + \frac{(1-i)}{2} x_2 = 0.$$

Hence, an eigenvector corresponding to $\lambda = -2 + i$ is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}.$$

Using the theorem above, an eigenvector corresponding to the eigenvalue $\overline{\lambda} = -2 - i$ is

$$\overline{\mathbf{x}} = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}.$$

Therefore, a solution is

$$\mathbf{x}(t) = e^{(-2+i)t} \begin{pmatrix} -1 \\ 1+i \end{pmatrix},$$

We decompose this vector into real and imaginary parts:

$$\begin{aligned}
\mathbf{x}(t) &= e^{(-2+i)t} \begin{pmatrix} -1 \\ 1+i \end{pmatrix} \\
&= e^{-2t}(\cos t + i \sin t) \begin{pmatrix} -1 \\ 1+i \end{pmatrix} \\
&= e^{-2t}(\cos t + i \sin t) \begin{pmatrix} -1 \\ 1+i \end{pmatrix} \\
&= \begin{pmatrix} -e^{-2t}(\cos t + i \sin t) \\ \\ e^{-2t}(\cos t + i \sin t)(1+i) \end{pmatrix}.
\end{aligned}$$

The real part (using $i^2 = -1$) is

$$\begin{aligned}
\begin{pmatrix} -e^{-2t} \cos t \\ e^{-2t} \cos t - e^{-2t} \sin t \end{pmatrix} &= \begin{pmatrix} -e^{-2t} \cos t \\ e^{-2t} \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ -e^{-2t} \sin t \end{pmatrix} \\
&= e^{-2t} \cos t \begin{pmatrix} -1 \\ 1 \end{pmatrix} - e^{-2t} \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

The imaginary part is

$$\begin{pmatrix} -e^{-2t} \sin t \\ e^{-2t} \sin t + e^{-2t} \cos t \end{pmatrix} = e^{-2t} \sin t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + e^{-2t} \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The vectors we obtained from the real and imaginary parts are linearly independent real solutions to the original differential equation.

This is true in general.

To find two linearly independent real-valued vector solutions to the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x},$$

corresponding to the complex eigenvalues $\lambda = \alpha \pm i\beta$ of $\mathbf{A}$, find the real and imaginary parts of the complex-valued vector solution to the system.

Solutions to linear systems have important geometric significance. In the 2×2 case, we can view the parametric plot of the solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

by plotting

$$(x_1(t), x_2(t))$$

on the xy-plane, for different values of t.

Such plots are called **parametric plots** of the solution.

If we plot many solutions with different initial values, we obtain a **phase portrait** for the system.

The eigenvalues of $\mathbf{A}$ tell us what kind of phase portrait results from the system.