Homogeneous Linear Systems with Constant Coefficients

Recall that a general system of first order linear differential equations has the form

$$
\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t)\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t)\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}
$$

We represent this system in matrix form by

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$

or, equivalently,

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ldots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

When the last vector $\mathbf{g}(t) = \mathbf{0}$, the system is said to be **homogeneous**. In this case, we have

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}.$$

In this lecture, we consider first order homogeneous systems where the functions $p_{ij}(t)$ are all constant functions.

Example. *Solve the first order system*

$$x_1' = x_1 - 3x_2$$
$$x_2' = -2x_1 + 2x_2.$$

We first write this system in matrix form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \Longrightarrow \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Note the resemblance of this differential equation to the exponential differential equation

$$y' = ry \qquad \Longrightarrow \qquad y(t) = Ke^{rt}.$$

Let's proceed as we did in chapter 3, but this time with vectors and matrices.

Let's try to find a solution $\mathbf{x}$ to the system of differential equations of the form

$$\mathbf{x} = e^{\lambda t} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where ξ_1 and ξ_2 are numbers.

Under this assumption,

$$\mathbf{x}' = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \lambda e^{\lambda t}.$$

Therefore, in order to satisfy the differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \Longrightarrow \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

we need to have

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \lambda e^{\lambda t} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}.$$

Subtracting the vector on the left side from both sides of the equation and factoring out the scalar function $e^{\lambda t}$, we get

$$\left[\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \lambda \right] e^{\lambda t} = 0.$$

By factoring and dividing both sides by $e^{\lambda t}$, we get

$$\left[\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

This last equation is equivalent to the requirement that λ is an eigenvalue for the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix}$$

and that $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is a corresponding eigenvector.

To find the eigenvalues for the matrix **A**, we first solve

$$\det(\mathbf{A}-\lambda I) = 0 \qquad \Longrightarrow \qquad \det\left[\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = 0.$$

This leads to

$$\det\begin{pmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{pmatrix} = 0 \qquad \Longrightarrow \qquad (1-\lambda)(2-\lambda)-6 = 0.$$

Therefore,

$$\lambda^2 - 3\lambda - 4 = 0 \qquad \Longrightarrow \qquad (\lambda+1)(\lambda-4) = 0.$$

Thus, the eigenvalues for **A** are $\lambda = -1$ and $\lambda = 4$.

To find the corresponding eigenvectors, we take each of these eigenvalues, plug them into

$$\left[\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0$$

to find the eigenvectors $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$.

For the eigenvalue $\lambda = -1$, we get

$$\left[\begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

By simplifying, we obtain

$$\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

To solve this system, we need to row reduce the corresponding augmented matrix

$$\begin{pmatrix} 2 & -3 & 0 \\ -2 & 3 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solution $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ satisfies

$$\begin{aligned} \xi_1 - \frac{3}{2}\xi_2 &= 0 \\ 0\xi_1 + 0\xi_2 &= 0. \end{aligned}$$

In other words, $\xi_1 = \frac{3}{2}\xi_2$ and ξ_2 is arbitrary. Choosing $\xi_2 = 2$, we see that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \xi_2 \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

is an eigenvector corresponding to the eigenvalue $\lambda = -1$.

By repeating this procedure for the other eigenvalue $\lambda = 4$, we find that

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvalue corresponding to $\lambda = 4$.

We assumed that $\mathbf{x}$ had the form

$$\mathbf{x} = e^{\lambda t} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where ξ_1 and ξ_2 were numbers.

We have now found each λ and corresponding vector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ that satisfy the differential equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \qquad \Longrightarrow \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

From the eigenvalue $\lambda = -1$, with associated eigenvector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, we obtain the solution

$$\mathbf{x} = e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

From the eigenvalue $\lambda = -4$ and corresponding eigenvector $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we obtain the solution

$$\mathbf{x} = e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The Wronskian of the two solutions is

$$\begin{vmatrix} 3e^{-t} & e^{4t} \\ 2e^{-t} & -e^{4t} \end{vmatrix} = -3e^{3t} - 2e^{3t} = -5e^{3t}.$$

Therefore, the solutions

$$e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

are linearly independent on $(-\infty, \infty)$.
Recall that this means

1. $$e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad e^{-t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

   form a fundamental set of solutions.

2. Any solution $\mathbf{x}(t)$ to the differential equation can be written

   $$\mathbf{x}(t) = c_1\begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix} + c_2\begin{pmatrix} 3e^{-t} \\ 2e^{-t} \end{pmatrix},$$

   for some constants c_1 and c_2.

Example. *Solve initial value problem*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

From the last example we found the general solution to be

$$\mathbf{x}(t) = c_1 \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix} + c_2 \begin{pmatrix} 3e^{-t} \\ 2e^{-t} \end{pmatrix},$$

for constants c_1 and c_2.

Then

$$\mathbf{x}(0) = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 \end{pmatrix} + \begin{pmatrix} 3c_2 \\ 2c_2 \end{pmatrix}$$

We need to solve the system

$$\begin{aligned} c_1 + 3c_2 &= 7, \\ -c_1 + 2c_2 &= 3. \end{aligned}$$

We row reduce the corresponding augmented matrix:

$$\begin{pmatrix} 1 & 3 & 7 \\ -1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & 5 & 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 7 \\ 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Therefore, $c_1 = 1$ and $c_2 = 2$, so the solution to the IVP is

$$\mathbf{x}(t) = \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix} + 2 \begin{pmatrix} 3e^{-t} \\ 2e^{-t} \end{pmatrix}.$$