Systems of First Order Linear Differential Equations

A general system of first order linear differential equations has the form

$$\begin{aligned}
x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t)\\
x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t)\\
&\vdots\\
x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t)
\end{aligned}$$

We represent this system in matrix form by

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$

or, equivalently,

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}.$$

When the last vector $\mathbf{g}(t) = \mathbf{0}$, the system is said to be **homogeneous**. In this case, we have

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}.$$

What does it mean for a vector to be a solution to a differential equation?

Example: The vectors

$$x^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \qquad x^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

are solutions to the equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

since

$$\frac{d}{dt}\mathbf{x}^{(1)} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix}.$$

Also,

$$\frac{d}{dt} \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ 2e^{-t} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

Thus, the derivative of the vector $\mathbf{x}^{(1)}$ equals the product

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}^{(1)},$$

and similarly for $\mathbf{x}^{(2)}$.

Suppose, in general, that

$$x^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \cdots, x^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}$$

are solutions to

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

We have the following **superposition principle** for first order systems of linear differential equations.

Theorem. *If the vector solutions* $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *are solutions to*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$

then the linear combination

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)}$$

is also a solution for any constants c_1 *and* c_2*.*

By iterating this theorem, we deduce that, if $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)}$ are solutions to

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x},$$

then so is

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$$

for any constants $c_1, c_2, \dots, c_n$.

Theorem. *If the vector functions* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ *are linearly independent solutions of the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$$

for each point in the interval $\alpha < t < \beta$, *then each solution* $\mathbf{x}$ *of the system can be expressed as a linear combination of* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$, *say,*

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$$

in exactly one way.

The linear combination of solutions in the last theorem is called the **general solution** to the system of linear equations.

The vector functions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are a **fundamental set of solutions** to the system of differential equations

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}.$$

The **Wronskian** of a set of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is the determinant of the matrix $(\mathbf{x}^{(1)} \mid \mathbf{x}^{(2)} \mid \ldots \mid \mathbf{x}^{(n)})$.

The next theorem says that, provided $\mathbf{P}(t)$ consists of continuous functions, the Wronskian of a set of solutions is either always zero or never zero.

Theorem. *If* $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ *are solutions of* $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ *on the interval* $\alpha < t < \beta$ *and* $\mathbf{P}(t)$ *is a matrix of continuous functions on* $\alpha < t < \beta$*, then the Wronskian,*

$$W(t) = \det(\mathbf{x}^{(1)}(t) \mid \mathbf{x}^{(2)}(t) \mid \ldots \mid \mathbf{x}^{(n)}(t)),$$

is either always zero or never zero.

Recall that the set of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is linearly independent at a point t_0 if

$$\det(\mathbf{x}^{(1)}(t_0) \mid \mathbf{x}^{(2)}(t_0) \mid \ldots \mid \mathbf{x}^{(n)}(t_0)) \neq 0.$$

If this determinant is zero, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are linearly dependent at t_0.

The last theorem says that, provided $\mathbf{P}(t)$ consists of continuous functions on a given interval, then

- A set of solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is a fundamental set of solutions if, at some point t_0 in the interval,

  $$W(t_0) = \det(\mathbf{x}^{(1)}(t_0) \mid \mathbf{x}^{(2)}(t_0) \mid \ldots \mid \mathbf{x}^{(n)}(t_0)) \neq 0.$$

- If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ are solutions to an equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, and the Wronskian equals zero at some point t_0, but is not identically zero, then at least one of the component functions of the matrix $\mathbf{P}(t)$ is discontinuous at t_0.

Example: The Wronskian of the solutions

$$x^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \qquad x^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

to the equation

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

equals

$$W(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t}.$$

Since $W(t)$ is not identically zero, the two solutions

$$x^{(1)}(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \qquad x^{(2)}(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$$

form a fundamental set of solutions.

That is, any solution to

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

can be written as a linear combination

$$c_1 x^{(1)}(t) + c_2 x^{(2)}(t).$$

Example: Suppose

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix}, \qquad x^{(2)}(t) = \begin{pmatrix} 5 \\ 3t \end{pmatrix}$$

are solutions to the system $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, for some $\mathbf{P}(t)$. The Wronskian of these solutions is

$$W(t) = \begin{vmatrix} 1 & 5 \\ 2t & 3t \end{vmatrix} = 3t - 10t = -7t.$$

Since the Wronskian is nonzero on $(-\infty, 0)$ and $(0, \infty)$, the vectors $x^{(1)}(t), x^{(2)}(t)$

- Are Linearly independent on $(-\infty, 0)$ and on $(0, \infty)$.
- Form a fundamental set of solutions on $(-\infty, 0)$ and on $(0, \infty)$.

Moreover, since the Wronskian is zero at $t = 0$, at least one of the components of $\mathbf{P}(t)$ is discontinuous at $t = 0$.