Ch 7.3: Sys. Lin. Eq., Lin. Ind., Eigenvalues

- A system of *n* linear equations in *n* variables,

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n\,, \end{aligned}$$

can be expressed as a matrix equation **Ax** = **b**:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

To see that the matrix equation is equivalent to the system of equations, multiply the matrix and vector on the left side and equate components.

- If **b** = **0**, the system is said to be **homogeneous**

Nonsingular (i.e., Invertible) Case

- An invertible matrix **A** is called nonsingular.
- If **A** is invertible, we can always solve **Ax** = **b** as follows:

$$\mathbf{Ax}=\mathbf{b} \;\Rightarrow \mathbf{A}^{-1}\mathbf{Ax}=\mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{Ix}=\mathbf{A}^{-1}\mathbf{b} \;\Rightarrow \mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$$

- This solution is unique.

- If **A** is invertible, the only solution to **Ax** = **0** is the trivial solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0}.$$

Example 1: Nonsingular Case (1 of 3)

- Here is an invertible matrix **A** and its inverse:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$$

- By multiplying by the inverse of **A** on both sides of the equation **Ax** = **0**, we obtain

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, the only solution to **Ax = 0** is **x = 0.**

Example 1: Nonsingular Case (2 of 2)

- Now let's solve the nonhomogeneous linear system **Ax** = **b** below using $\mathbf{A}^{-1}$:

$$\begin{aligned} 0x_1 + x_2 + 2x_3 &= \ \ 2 \\ 1x_1 + 0x_2 + 3x_3 &= -2 \\ 4x_1 - 3x_2 + 8x_3 &= \ \ 0 \end{aligned}$$

- This system of equations can be written as **Ax** = **b**, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$$

- Then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -23 \\ -12 \\ 7 \end{pmatrix}$$

Singular Case

- If the coefficient matrix **A** is singular (not invertible), then either there is no solution to **Ax** = **b**, or there are infinitely many solutions to **Ax** = **b**.

We can find these solutions by row reducing **(A | b).**

Linear Dependence and Independence

- A set of vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is **linearly dependent** if there exists scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)} = \mathbf{0}$$

- If the only solution of

  is $c_1 = c_2 = \ldots = c_n = 0$, then $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n)}$ is **linearly independent.**

Example 3: Linear Independence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix}$$

- We need to determine for what coefficients

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + c_3\begin{pmatrix} 2 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 3: Linear Independence (2 of 2)

- We reduce the augmented matrix (**A**|**b**).

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 4 & -3 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\rightarrow \begin{matrix} c_1 & & + 3c_3 & = 0 \\ & c_2 & + 2c_3 & = 0 \\ & & c_3 & = 0 \end{matrix} \rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Thus, the only solution is $c_1 = c_2 = \ldots = c_n = 0$, and therefore the original vectors are linearly independent.

Example 4: Linear Dependence (1 of 2)

- Determine whether the following vectors are linear dependent or linearly independent.

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix}, \quad \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$$

- We need to solve

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + c_3\mathbf{x}^{(3)} = \mathbf{0}$$

or

$$c_1\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} + c_3\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Example 4: Linear Dependence (2 of 2)

- We thus reduce the augmented matrix (**A**|**b**), as before.

$$(\mathbf{A}|\mathbf{b}) = \begin{pmatrix} 1 & -2 & -1 & 0 \\ -1 & 5 & 6 & 0 \\ 5 & -4 & 5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{array}{rrrl} c_1 & -2c_2 & -1c_3 & =0 \\ & 3c_2 & +5c_3 & =0 \\ & & 0c_3 & =0 \end{array} \to \mathbf{c} = \begin{pmatrix} -7c_3/3 \\ -5c_3/3 \\ c_3 \end{pmatrix} \to \mathbf{c} = k\begin{pmatrix} 7 \\ 5 \\ -3 \end{pmatrix}$$

Thus the original vectors are linearly dependent, with

$$7\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + 5\begin{pmatrix} -2 \\ 5 \\ -4 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Linear Independence and Invertibility

- Consider the previous two examples:
  - The first matrix was known to be nonsingular, and its column vectors were linearly independent.
  - The second matrix was known to be singular, and its column vectors were linearly dependent.
- This is true in general: the columns (or rows) of $\mathbf{A}$ are linearly independent if and only if $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}^{-1}$ exists.
- Also, $\mathbf{A}$ is nonsingular if and only if $\det \mathbf{A} \neq 0$, hence columns (or rows) of $\mathbf{A}$ are linearly independent if and only if $\det \mathbf{A} \neq 0$.

Linear Dependence & Vector Functions

- Now consider vector functions $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,…, $\mathbf{x}^{(n)}(t)$, where

$$\mathbf{x}^{(k)}(t) = \begin{pmatrix} x_1^{(k)}(t) \\ x_2^{(k)}(t) \\ \vdots \\ x_m^{(k)}(t) \end{pmatrix}, \; k = 1, 2, \ldots, n, \qquad t \in I = (\alpha, \beta)$$

- As before, $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,…, $\mathbf{x}^{(n)}(t)$ is **linearly dependent** on I if there exists scalars c_1, c_2,…, c_n, not all zero, such that

$$c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) = \mathbf{0}, \; \text{for all } t \in I$$

- Otherwise $\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$,…, $\mathbf{x}^{(n)}(t)$ is **linearly independent** on I

Eigenvalues and Eigenvectors

- The eqn. $\mathbf{Ax} = \mathbf{y}$ can be viewed as a map of the vector $\mathbf{x}$ into a new vector $\mathbf{y}$.
- Nonzero vectors $\mathbf{x}$ that $\mathbf{A}$ maps to multiples of themselves are important in applications.
- *We* will be interested in finding numbers λ and corresponding vectors $\mathbf{x}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$ or equivalently, $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.
- This equation has a nonzero solution if we choose λ such that $\det(\mathbf{A}-\lambda\mathbf{I}) = 0$.
- Such values of λ are called **eigenvalues** of **A**, and the corresponding nonzero solutions **x** are called **eigenvectors**.

Example 5: Eigenvalues (1 of 3)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det\left(\begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix}$$

$$= (2-\lambda)(-6-\lambda) - (3)(3)$$

$$= \lambda^2 + 4\lambda - 21 = (\lambda - 3)(\lambda + 7)$$

$$\Rightarrow \lambda = 3, \ \lambda = -7$$

Example 5: First Eigenvector (2 of 3)

- To find the eigenvectors of the matrix **A**, we need to solve (**A**-λ**I**)**x** = **0** for λ = 3 and λ = -7.
- **<u>Eigenvector for λ = 3:</u>** Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \iff \begin{pmatrix} 2-3 & 3 \\ 3 & -6-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} -1 & 3 & 0 \\ 3 & -9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 3 & -9 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrl} 1x_1 & -3x_2 & =0 \\ & 0x_2 & =0 \end{array}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} 3x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 3 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Example 5: Second Eigenvector (3 of 3)

- **Eigenvector for λ = -7:** Solve

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{x}=\mathbf{0} \Leftrightarrow \begin{pmatrix} 2+7 & 3 \\ 3 & -6+7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by row reducing the augmented matrix:

$$\begin{pmatrix} 9 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/3 & 0 \\ 3 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{matrix} 1x_1 & +1/3x_2 & =0 \\ & 0x_2 & =0 \end{matrix}$$

$$\to \mathbf{x}^{(2)} = \begin{pmatrix} -1/3x_2 \\ x_2 \end{pmatrix} = c\begin{pmatrix} -1/3 \\ 1 \end{pmatrix},\ c \text{ arbitrary} \to \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Normalized Eigenvectors

- From the previous example, we see that eigenvectors are determined up to a nonzero multiplicative constant.
- If this constant is specified in some particular way, then the eigenvector is said to be **normalized**.
- For example, eigenvectors are sometimes normalized by choosing the constant so that $||\mathbf{x}|| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$.

Algebraic and Geometric Multiplicity

- In finding the eigenvalues λ of an n x n matrix **A**, we solve det(**A**-λ**I**) = 0.
- Since this involves finding the determinant of an n x n matrix, the problem reduces to finding roots of an nth degree polynomial.
- Denote these roots, or eigenvalues, by $\lambda_1, \lambda_2, \ldots, \lambda_n$.
- If an eigenvalue is repeated m times, then its **algebraic multiplicity** is m.
- Each eigenvalue has at least one eigenvector, and a eigenvalue of algebraic multiplicity m may have q linearly independent eigenvectors, $1 \leq q \leq m$, where q is called the **geometric multiplicity** of the eigenvalue.

Eigenvectors and Linear Independence

- If an eigenvalue λ has algebraic multiplicity 1, then it is said to be **simple**, and the geometric multiplicity is 1 also.
- If each eigenvalue of an *n* x *n* matrix **A** is **simple**, then **A** has *n* distinct eigenvalues. It can be shown that the *n* eigenvectors corresponding to these eigenvalues are linearly independent.
- If an eigenvalue has one or more **repeated eigenvalues**, then there may be fewer than *n* linearly independent eigenvectors. This may lead to complications in solving systems of differential equations.

Example 6: Eigenvalues (1 of 5)

- Find the eigenvalues and eigenvectors of the matrix **A**.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- Solution: Choose λ such that det(**A**-λ**I**) = 0, as follows.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 + 3\lambda + 2$$

$$= (\lambda - 2)(\lambda + 1)^2$$

$$\Rightarrow \lambda_1 = 2, \; \lambda_2 = -1, \; \lambda_3 = -1$$

Example 6: First Eigenvector (2 of 5)

- Eigenvector for $\lambda = 2$: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{matrix} 1x_1 & & -1x_3 & = 0 \\ & 1x_2 & -1x_3 & = 0 \\ & & 0x_3 & = 0 \end{matrix}$$

$$\rightarrow \mathbf{x}^{(1)} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; c \text{ arbitrary} \rightarrow \text{choose } \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Example 6: 2nd and 3rd Eigenvectors (3 of 5)

- Eigenvector for λ = -1: Solve $(\mathbf{A}-\lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, as follows.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{rrrl} 1x_1 & +1x_2 & +1x_3 & =0 \\ & 0x_2 & & =0 \\ & & 0x_3 & =0 \end{array}$$

$$\to \mathbf{x}^{(1)} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{ where } x_2, x_3 \text{ arbitrary}$$

$$\to \text{choose } \mathbf{x}^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Example 6: Eigenvectors of **A** (4 of 5)

- Thus three eigenvectors of **A** are

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{x}^{(2)} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

  where $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ correspond to the double eigenvalue λ = - 1.

- It can be shown that $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$ are linearly independent.
- Hence **A** is a 3 x 3 **symmetric matrix** (**A** = $\mathbf{A}^T$) with 3 real eigenvalues and 3 linearly independent eigenvectors.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Example 6: Eigenvectors of **A** (5 of 5)

The eigenvectors in the last example are orthogonal, since

$$\left(\mathbf{x}^{(1)},\mathbf{x}^{(2)}\right)=0,\quad \left(\mathbf{x}^{(1)},\mathbf{x}^{(3)}\right)=0,\quad \left(\mathbf{x}^{(2)},\mathbf{x}^{(3)}\right)=0$$

- Thus **A** is a 3 x 3 symmetric matrix with 3 real eigenvalues and 3 linearly independent orthogonal eigenvectors.
- We could have also chosen

$$\mathbf{x}^{(2)}=\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \mathbf{x}^{(3)}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Hermitian Matrices

- A **self-adjoint**, or **Hermitian** matrix, satisfies $\mathbf{A} = \mathbf{A}^*$.
- Note that if **A** has real entries and is symmetric (see last example), then **A** is Hermitian.
- An n x n Hermitian matrix **A** has the following properties:
  - All eigenvalues of **A** are real.
  - There exists a full set of n linearly independent eigenvectors of **A**.
  - If $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are eigenvectors that correspond to different eigenvalues of **A**, then $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are orthogonal.
  - Corresponding to an eigenvalue of algebraic multiplicity m, it is possible to choose m mutually orthogonal eigenvectors, and hence A has a full set of n linearly independent orthogonal eigenvectors.