Systems of Differential Equations

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Systems of Differential Equations

In many applications, two or more interrelated rates of change (derivatives) are at work in a process that we want to model.

In these situations, we obtain two or more simultaneous differential equations that must be satisfied at the same time.

Such systems can be studied effectively using linear algebra.

The Predator-Prey Model

Suppose we want to model the populations in a given predator prey ecosystem.

Let $F(t)$ denote the population of foxes in an ecosystem at time t, and let $R(t)$ denote the number of Rabbits at time t.

The predator population is dependent upon the prey population (foxes need to eat) and the prey population is dependent upon the predator population (rabbits need to live).

A mathematical model for this system is

$$\frac{dF}{dt} = -a_1 F + b_1 FR, \qquad \frac{dR}{dt} = a_2 R - b_2 FR.$$

The Predator-Prey model

For positive values of a_1, a_2, b_1, b_2, why does the system

$$\frac{dF}{dt} = -a_1 F + b_1 FR, \qquad \frac{dR}{dt} = a_2 F - b_2 FR.$$

describe the predator-prey model?

- In the absence of rabbits ($R(t) = 0$), the population of foxes would decay exponentially (we would have $\frac{dF}{dt} = -a_1 F$).
- A higher rabbit population is good for the fox population, so a larger $R(t)$ leads to larger growth rate for the foxes ($\frac{dF}{dt}$). The likelihood of interaction between the species is proportional to the size of each population.
- The differential equation for $R(t)$ can be similarly motivated.

Other Differential Equations

Systems of differential equations arise in many applications in engineering, physics, and pure mathematics:

- In electrical engineering to describe complicated circuits.
- In physics to describe situations where multiple forces are at work.
- In number theory and other areas of mathematics to describe interrelated functions.
- Many numerical algorithms for solving differential equations require that the input be a first order system.

Writing Higher order DEs as Systems of DEs

Higher order differential equations can always be written as a system of first order differential equations.

Consider the spring-mass system we analyzed in Chapter 3:

$$mu'' + \gamma u' + ku = F(t).$$

Set

$$x_1(t) = u(t), \qquad x_2(t) = u'(t).$$

The differential equation can be written

$$mx_2' + \gamma x_2 + kx_1 = F(t).$$

All together, we have

$$x_1' = x_2, \qquad x_2' = \frac{-k}{m}x_1 - \frac{\gamma}{m}x_2 + \frac{F}{m}.$$

Writing Higher order DEs as Systems of DEs

In general, an arbitrary *n*th order differential equation

$$y^{(n)} = F(t, y, y', y'', \dots, y^{(n-1)})$$

can be written, via

$$x_1 = y, \qquad x_2 = y', \qquad x_3 = y'', \quad \dots, \quad x_n = y^{(n-1)}$$

as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= F(t, x_1, x_2, x_3, \dots, x_n). \end{aligned}$$

Solutions of Systems of Differential Equations

A general system of n differential equations can be written:

$$\begin{aligned} x_1' &= F_1(t, x_1, x_2, \ldots, x_n), \\ x_2' &= F_2(t, x_1, x_2, \ldots, x_n), \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \ldots, x_n), \end{aligned}$$

a **solution** is a set of functions

$$x_1(t), x_2(t), x_3(t), \ldots, x_n(t)$$

that each satisfy the differential equations. If we also require

$$x_1(t_0) = \alpha_1, \quad x_2(t_0) = \alpha_2, \quad x_3(t_0) = \alpha_3, \quad \ldots, \quad x_n(t_0) = \alpha_n,$$

we arrive at an **initial value problem** for the system.

Solutions to systems viewed as Parametric Equations

Consider the differential equation

$$y'' + y = 0, \qquad 0 < t < 2\pi.$$

We first convert this to a system via

$$x_1 = y, \qquad x_2 = y'.$$

Then the differential equation can be written

$$x_2' + x_1 = 0.$$

So, all together, we have the system

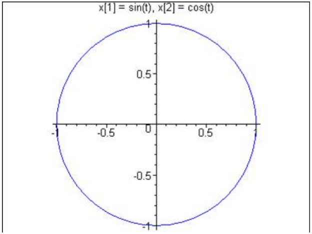
$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1. \end{aligned}$$

Solutions to systems viewed as Parametric Equations

How do we find functions $x_1(t)$ and $x_2(t)$ that are interrelated by

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -x_1 \qquad ? \end{aligned}$$

The pair $x_1(t) = \sin t$, $x_2 = \cos t$ works. What do we get if we plot all the points $(x_1(t), x_2(t))$ on the xy-plane for $0 < t < 2\pi$?



The pair $x_1(t) = \sin t$, $x_2(t) = \cos t$ form a **parametric representation** for the unit circle.