Solving Differential Equations with The Laplace Transform

The usefulness of the Laplace transform in solving initial value problems comes from the fact that $\mathcal{L}\{f'(t)\}$ is related to $\mathcal{L}\{f(t)\}$.

Theorem. *Suppose that f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Suppose futher that there exist constants K, a, and M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f(t)\}$ exists for $s > a$, and moreover*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

Proof Suppose that the points of discontinuity of f on $0 \le t \le A$ are

$$0 \le t_1 < t_2 < \cdots < t_n \le A.$$

Then, integrating over each subinterval,

$$\int_0^A e^{-st} f'(t)\ dt = \int_0^{t_1} e^{-st} f'(t)\ dt + \int_{t_1}^{t_2} e^{-st} f'(t)\ dt + \cdots + \int_{t_n}^{A} e^{-st} f'(t)\ dt$$

We evaluate each integral using integration by parts:

$$u = e^{-st}, \qquad dv = f'(t)\,dt, \qquad v = f(t), \qquad du = -se^{-st}\,dt.$$

$$\begin{aligned}
&\int_0^A e^{-st} f'(t)\ dt \\
&\qquad = \left\{e^{-st} f(t)\right\}_{t=0}^{t=t_1} + \left\{e^{-st} f(t)\right\}_{t=t_1}^{t=t_2} + \cdots + \left\{e^{-st} f(t)\right\}_{t=t_n}^{t=A} \\
&\qquad\qquad + s \int_0^{t_1} e^{-st} f(t)\ dt + s \int_{t_1}^{t_2} e^{-st} f(t)\ dt \\
&\qquad\qquad\quad + \cdots + s \int_{t_n}^{A} e^{-st} f(t)\ dt \\
&\qquad = \left(e^{-st_1} f(t_1) - f(0)\right) + \left(e^{-st_2} f(t_2) - e^{-st_1} f(t_1)\right) \\
&\qquad\quad + \cdots + \left(e^{-sA} f(A) - e^{-st_n} f(t_n)\right) + s \int_0^A f(t) e^{-st}\ dt \\
&\qquad = e^{-sA} f(A) - f(0) + s \int_0^A f(t) e^{-st}\ dt.
\end{aligned}$$

- The terms in parentheses above “telescoped” – or cancelled each other out successively, leaving us with only the first and last terms.
- To get the integral

  $$s \int_0^A f(t) e^{-st}\ dt,$$

  the corresponding integrals over subintervals were re-combined to form a single integral over the entire interval.

So, all together, we have

$$\int_0^A e^{-st} f'(t)\ dt = e^{-sA} f(A) - f(0) + s \int_0^A f(t) e^{-st}\ dt.$$

Taking the limit as $A \to \infty$ on each side, we obtain

$$\int_0^\infty e^{-st} f'(t)\ dt = 0 - f(0) + s \int_0^\infty f(t) e^{-st}\ dt.$$

Equivalently,

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}.$$

□

Since the second derivative is the derivative of the derivative, the same formula says that

$$\begin{aligned} \mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0) \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0). \end{aligned}$$

In general, by iterating the relation for higher order derivatives, we get

$$\begin{aligned} \mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) + s^{n-2}f'(0) \\ - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0). \end{aligned}$$

This last identity is only valid under certain assumptions.

Theorem. *Suppose that the functions* $f(t), f'(t), \ldots, f^{(n-1)}(t)$ *are continuous and that* $f^{(n)}(t)$ *is piecewise continuous on any interval* $0 \leq t \leq A$*. Suppose further that there exist constants* $0 \leq t \leq A$ *such that, for* $t \geq M$*,*

$$|f(t)| \leq Ke^{at}, \quad |f'(t)| \leq Ke^{at}, \quad \cdots, \quad |f^{(n-1)}(t)| \leq Ke^{at}.$$

Then $\mathcal{L}\{f(t)\}$ *exists, and*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) + s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Example. *Use the Laplace transform to solve the initial value problem*

$$y'' + 6y' - 7y = 0,$$

$$y(0) = 1, \qquad y'(0) = 0.$$

Clearly we could solve this by finding the roots of the characteristic equation, but let us see how the Laplace transform can lead us to the same solution.

Since the Laplace transform is linear, the differential equation implies

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} - 7\mathcal{L}\{y\} = 0. \tag{1}$$

Using the recursion formula from the theorem above, we may write the left side of (1) in terms of $\mathcal{L}\{f(t)\}$.

Applying the above theorem, we get

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 6[s\mathcal{L}\{y\} - y(0)] - 7\mathcal{L}\{y\} = 0.$$

Combining terms with $\mathcal{L}\{y\}$ and combining terms with $y(0)$ gives us

$$(s^2 + 6s - 7)\mathcal{L}\{y\} - (6 + s)y(0) - y'(0) = 0.$$

Plugging in the initial conditions,

$$y(0) = 1, \qquad y'(0) = 0,$$

we get

$$(s^2 + 6s - 7)\mathcal{L}\{y\} - (6 + s) = 0 \qquad \Longrightarrow \qquad \mathcal{L}\{y\} = \frac{6 + s}{s^2 + 6s - 7}.$$

Factoring the denominator, we expand by partial fractions

$$\mathcal{L}\{y\} = \frac{6 + s}{(s - 1)(s + 7)} = \frac{A}{s - 1} + \frac{B}{s + 7}.$$

To find the numbers A and B, multiply both sides of the last equality by $(s - 1)(s + 7)$ to get

$$6 + s = A(s + 7) + B(s - 1) = (A + B)s + 7A - B.$$

Therefore, equating coefficients on both sides, we find that

$$A + B = 1, \qquad \text{and} \qquad 7A - B = 6.$$

Solving this system results in

$$A = \frac{7}{8}, \qquad B = \frac{1}{8}.$$

This means that

$$\mathcal{L}\{y\} = \frac{7/8}{s-1} + \frac{1/8}{s+7}.$$

Thus, to find $y(t)$, we need to find the functions whose Laplace transforms are

$$\frac{7/8}{s-1} \qquad \text{and} \qquad \frac{1/8}{s+7}.$$

We showed yesterday that

$$\int_0^\infty e^{ct}\, dt = \quad = \begin{cases} -1/c, & \text{if } c < 0, \\ \text{The integral diverges,} & \text{if } c \geq 0. \end{cases}$$

Therefore,

$$\int_0^\infty e^{-st} e^{t}\, dt = \int_0^\infty e^{(-s+1)t}\, dt = \frac{1}{s-1}, \qquad s > 1.$$

and

$$\int_0^\infty e^{-st} e^{-7t}\, dt = \int_0^\infty e^{(-s-7)t}\, dt = \frac{1}{s+7}, \qquad s > -7.$$

We conclude that,

$$\mathcal{L}\{e^{t}\} = \frac{1}{s-1}, \qquad \mathcal{L}\{e^{-7t}\} = \frac{1}{s+7}.$$

From these calculations, and by the linearity of the Laplace transform,

$$\mathcal{L}\{y\} = \frac{7/8}{s-1} + \frac{1/8}{s+7} \qquad \Longrightarrow \qquad y(t) = \frac{7}{8}e^{t} + \frac{1}{8}e^{-7t}.$$

In general, we often want to know how to “undo” a Laplace transform.

That is, given a Laplace transform $F(s)$, what function $f(t)$ is such that $\mathcal{L}\{f(t)\} = F(s)$?

If we denote the **inverse Laplace transform** of $F(s)$ by $\mathcal{L}^{-1}\{F(s)\}$, then we look for an $f(t)$ so that

$$\mathcal{L}^{-1}\{F(s)\} = f(t).$$

If we can't easily compute $\mathcal{L}^{-1}\{F(s)\}$ by hand, there are many references for common inverse Laplace transforms (See p. 319 of our textbook).

Computer algebra systems can also compute many inverse Laplace transforms.

Just as the Laplace transform is a linear operator, the inverse of the Laplace transform is also linear:

If

$$F(s) = c_1 F_1(s) + c_2 F_2(s) + \cdots + c_n F_n(s),$$

then

$$\mathcal{L}^{-1}\{F(s)\} = c_1 \mathcal{L}^{-1}\{F_1(s)\} + c_2 \mathcal{L}^{-1}\{F_2(s)\} + \cdots + \mathcal{L}^{-1}\{F_n(s)\}.$$

This means that you can compute the inverse Laplace transform term-by-term, as we did in the last example.

Example. *Find the inverse Laplace transform of*

$$\frac{8s^2-4s+12}{s(s^2+4)}.$$

Notice that the denominator is factored completely over the reals. The partial fraction expansion is given by

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

This implies that

$$8s^2-4s+12 = A(s^2+4) + Bs^2 + Cs = (A+B)s^2 + Cs + 4A$$

By equating coefficients of each power of s, we find

$$A+B=8, \qquad C=-4, \qquad 4A=12.$$

Therefore, $\qquad A=3, \qquad B=5, \qquad C=-4.$

$$\frac{8s^2-4s+12}{s(s^2+4)} = \frac{3}{s} + \frac{5s-4}{s^2+4} = \frac{3}{s} + \frac{5s}{s^2+4} - \frac{4}{s^2+4}.$$

Hence, using the table on page 319 of our textbook,

$$\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = 3 + 5\cos 2t + 2\sin 2t.$$

Under appropriate conditions, we can differentiate the Laplace transform by differentiating under the integral sign:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\ dt = \int_0^\infty -te^{-st}f(t)\ dt = \mathcal{L}\{-tf(t)\}.$$

Differentiating again and again, we find that

$$\frac{d^2}{ds^2}F(s) = \mathcal{L}\{(-t)^2f(t)\}, \quad \cdots, \quad \frac{d^n}{ds^n}F(s) = \mathcal{L}\{(-t)^nf(t)\}.$$

Example. *Find the Laplace transform of* $te^{at}\sin bt$*.*

From the above calculations, and the linearity of the Laplace transform,

$$\mathcal{L}\{te^{at}\sin bt\} = \frac{d}{ds}\mathcal{L}\{-e^{at}\sin bt\} = -\frac{d}{ds}\mathcal{L}\{e^{at}\sin bt\}.$$

Entry 9 in our table (p. 319) implies that

$$\begin{aligned} -\frac{d}{ds}\mathcal{L}\{e^{at}\sin bt\} &= -\frac{d}{ds}\left(\frac{b}{(s-a)^2+b^2}\right) \\ &= b\left((s-a)^2+b^2\right)^{-2}(2s-2a). \end{aligned}$$

Therefore,

$$\mathcal{L}\{te^{at}\sin bt\} = \frac{2b(s-a)}{\left((s-a)^2+b^2\right)^2}.$$

The Laplace transform of a function involving $\pm t^n$ can often be computed by taking an iterated derivative (with respect to s) of the Laplace transform corresponding to a simpler function.